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A NOTE ON SYMMETRIC BERNOULLI RANDOM VARIABLES[1]

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Consider independent random variables $X_1, X_2, \cdots$ such that X_i takes the values ± 1 each with probability $\frac{1}{2}$. If $\theta = (\theta_1, \cdots, \theta_n)$ satisfies $\sum_{i=1}^n \theta_i^2 = 1$, let $S_n(\theta) = \sum_1^n \theta_i X_i$ and $S_n = n^{-\frac{1}{2}} \sum_1^n X_i$. Recently, Efron (1969) has shown that $E(S_n(\theta))^{2k} \leqq ES_n^{2k}$ for $k = 1, 2, \cdots$ and for all n. In the present note, sufficient conditions on a continuously differentiable function f are given so that $Ef(S_n(\theta)) \leqq Ef(S_n)$ for all n. This result is then used to derive probability bounds related to results of Hoeffding (1963).

Definition 1. Let $a = (a_1, \cdots, a_m)$ and $b = (b_1, \cdots, b_m)$ be real vectors. Reorder the components of a and b such that $a_{i_1} \geqq a_{i_2} \geqq \cdots \geqq a_{i_m}$ and $b_{j_1} \geqq b_{j_2} \geqq \cdots \geqq b_{j_m}$. Then, *a majorizes b* if and only if $\sum_{\alpha=1}^k a_{i_\alpha} \geqq \sum_{\alpha=1}^k b_{j_\alpha}$ for $k = 1, \cdots, m-1$ and $\sum_{\alpha=1}^m a_{i_\alpha} = \sum_{\alpha=1}^m b_{j_\alpha}$.

Definition 2. A real-valued function φ defined on an open subset of R^n which has continuous first partial derivatives is called a Schur function if

$$\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j} \geqq 0 \tag{1}$$

when $x_i > x_j$ for $i, j = 1, \cdots, n$ and x in the domain of φ.

A result which relates Schur functions and one vector majorizing another is

Theorem. *Let C be an open symmetric convex set in R^n and suppose φ is a Schur function on C which is a symmetric function of its arguments. If $a \in C$ majorizes $b \in C$, then $\varphi(a) \geqq \varphi(b)$.*

For a proof of this theorem, see Schur (1923) and Ostrowski (1952).

Now, let F be the set of all functions f on R to R which are continuously differentiable and satisfy

$$t^{-1}[f'(t+\Delta) - f'(-t+\Delta) + f'(t-\Delta) - f'(-t-\Delta)] \tag{2}$$

is non-decreasing in t for $t > 0$ and $\Delta \geqq 0$. Note that $f \in F$ is equivalent to

$$t^{-1} E_W[f'(t+W) - f'(-t+W)] \tag{3}$$

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