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LINEAR SPACES AND UNBIASED ESTIMATION[1]

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1. Introduction. In this paper some general results are obtained on unbiased estimation when the choice of estimators is restricted to a finite-dimensional linear space $\bar{\mathscr{A}}$. The results concern mainly necessary and sufficient conditions for existence of unbiased estimators within $\bar{\mathscr{A}}$ and procedures for obtaining such estimators when they exist. At the outset the approach is from a coordinate-free (Kruskal [2]) viewpoint; but then it is found useful, and for many situations natural, to use a fixed reference set of spanning elements for particular subspaces. Identical analogues with estimation procedures in what is commonly referred to as linear model theory will be seen to exist. Much of the formulation has been motivated by problems especially relevant in the study of a general mixed linear model $Y = X\beta + e$ where the random vector Y has expectation $X\beta$ and covariance matrix

$$E[ee'] = \sum_{i=1}^{m} \nu_i V_i.$$

Concerning this model attention is given in a second paper to quadratic estimation, i.e., $\bar{\mathscr{A}} = \{Y'AY: A \text{ real and symmetric}\}$, of parametric functions of the form $\sum_{i \leq j} \lambda_{ij}\beta_i\beta_j + \sum_k \lambda_k \nu_k$ with special emphasis on parametric functions of the form $\sum_k \lambda_k \nu_k$.

2. Preliminary framework. Terminology and elementary properties associated with linear spaces (vector spaces) are utilized in the sequel. Much of the notation and terminology is similar to that used in Chapter II of Wilansky [3] and in Halmos [1]. Since only finite-dimensional real linear spaces are considered, we adopt the convention that whenever a linear space is referred to it is assumed to be a finite-dimensional real linear space. In addition to vector space notions, the usual notations and ideas of elementary set theory are also employed.

Concerning vector space notions, we mention a few at this point. If $\mathscr{A}$ and $\mathscr{B}$ are non-empty subsets of a vector space $\mathscr{L}$, then $\mathscr{A} + \mathscr{B}$ denotes the set $\{a+b: a \in \mathscr{A}, b \in \mathscr{B}\}$. When the set $\mathscr{A}$ consists of a single element a, we abbreviate $\mathscr{A} + \mathscr{B}$ by $a + \mathscr{B}$. The linear span of a non-empty subset $\mathscr{A}$ of $\mathscr{L}$ is denoted by $\operatorname{sp}\mathscr{A}$. If $\mathscr{A}$ and $\mathscr{B}$ are disjoint subspaces of $\mathscr{L}$, i.e., their intersection is the null vector, then we denote the sum of $\mathscr{A}$ and $\mathscr{B}$ by $\mathscr{A} \oplus \mathscr{B}$. The null vector is denoted by 0, and even though the symbol 0 is used for other purposes, it should be clear from the context what 0 represents.

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