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A CLASS OF ORTHOGONAL SERIES RELATED TO MARTINGALES[1]

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1. Introduction. In this paper we study convergence problems for sums of dependent random variables. The particular type of series considered here includes all discrete parameter martingales, but is more restricted than the class of all orthogonal series.

Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of centered (mean zero) random variables on a probability space (Ω, F, P) and $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$ their sequence of partial sums. Consider the following hierarchy of dependence.

(1.1) Mutual independence:

$$\int_{\Omega}\Theta(X_1, X_2, \cdots, X_n)\Phi(X_{n+1})\,dP = \int_{\Omega}\Theta(X_1, X_2, \cdots, X_n)\,dP\int_{\Omega}\Phi(X_{n+1})\,dP$$

for all pairs $(\Theta(\cdot), \Phi(\cdot))$ of integrable functions of the indicated variables, $n = 1, 2, \cdots$.

(1.2) The *martingale* property: $\int_{\Omega}\Theta(X_1, X_2, \cdots, X_n)X_{n+1}\,dP = 0$

for all bounded measurable functions $\Theta(\cdot)$ of the indicated variables, $n = 1, 2, \cdots$.

(1.3) The *weak martingale* property: $\int_{\Omega}\Phi(Y_m)X_n\,dP = 0$

for all bounded measurable functions $\Phi(Y_m)$, $1 \leqq m < n = 2, 3, \cdots$.

(1.4) Orthogonality: $\int_{\Omega}X_nX_m\,dP = 0, \qquad n \neq m.$

If $X_mX_n \in L^1$ for $m \neq n$, the weak martingale property implies orthogonality (Proposition 2.2). Otherwise, the increasing dependence of the hierarchy is clear. If $\{Y_n\}_{n=1}^{\infty}$ satisfies (1.2), it is called a *martingale*. If $\{Y_n\}_{n=1}^{\infty}$ satisfies (1.3), we call it a *weak martingale*. Weak submartingales are defined analogously. Clearly, every martingale is a weak martingale and Gaussian weak martingales are martingales (Section 3).

Martingales which converge in L^2, converge a.e.; but, there are orthogonal series which converge in L^2 and diverge a.e. ([1] Theorem 2.4.1, page 88). Since weak martingales lie between orthogonal series and martingales in the hierarchy of dependence, it is natural to investigate the pointwise convergence of L^2—convergent weak martingales. We show that on totally finite signed measure spaces martingales whose L^2 norms are bounded converge a.e. (Theorem 4.1). But, we construct an example of an L^2-bounded weak martingale on a totally finite signed measure space whose paths oscillate between plus and minus infinity.

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