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# SOME REMARKS ON THE FELLER PROPERTY[1]

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**1. Introduction.** Let $X_t$ be a temporally homogeneous Markov process on a locally compact metric space $E$. Denote its transition probabilities by $P_t(x, B)$, $t \geqq 0$, $x \in E$, $B \in \mathscr{B}$, where $\mathscr{B}$ is the topological Borel field on $E$. Let $C_0$ and $\mathscr{B}_b$ be the set of continuous functions with limit zero at infinity, and the bounded Borel measurable functions respectively. The transition probabilities can be thought of as operators on $\mathscr{B}_b$ via the formula $P_t f(x) = \int_E P_t(x, dy) f(y)$. The process $X$ is said to be a Feller process if

(i) $P_t : C_0 \to C_0$,

(ii) for each $x \in E$ and $f \in C_0$, $P_t f(x) \to f(x)$ as $t \downarrow 0$.

If $X$ is a Feller process, we can always find a standard modification $\hat{X}$ of $X$ which is a Hunt process; that is, $\hat{X}$ is a right-continuous strong Markov process having left limits in $E$, and furthermore is quasi-left continuous: if $\{T_n\}$ is an increasing sequence of stopping times with limit $T$, then $\hat{X}(T_n) \to \hat{X}(T)$ w.p. 1. Most of the Markov processes presently admitted to the select circle of "well-behaved processes" —Brownian motion, for instance, or more generally most diffusion and birth-and-death processes—are Feller processes. On the other hand the Feller property is far from being a necessary condition that $X$ be a Hunt process, and its consequences are often consequences of the continuity properties of $s \to P_t f \circ X_s$ rather than of $x \to P_t f(x)$. For instance, it is an easy exercise to show that if $X$ is a right-continuous Markov process and $s \to P_t f \circ X_s$ is right continuous for each $f \in C_0$, then $X$ must be strongly Markov. In fact, this condition turns out to be both necessary and sufficient for $X$ to be strongly Markov (Theorem 2.1).

In this paper, we will show that every strong Markov process satisfies Feller properties of the second type—here we use the term "Feller property" very loosely to mean any relation of the type "$P_t$ takes a class of 'well-behaved' functions into another class of 'well-behaved' functions." Such properties can often be described topologically, though this is not always the most convenient way. If $X_t$ is a right-continuous strong Markov process, there is a topology on $E$, called the fine topology, which is particularly well adapted to the process. A set $B \subset E$ is open in the fine topology if and only if for each $x \in B$, $P^x\{X_t \in B$ for some interval $(0, \delta)\} = 1$. Girsanov [5] and Šur [10] proved that if $X_t$ is a Hunt process, $P_t$ takes the class of bounded fine continuous functions into itself. This is equivalent to the statement that $s \to P_t f \circ X_s$ is right continuous a.e. $(P^x)$ for each $x$ whenever $f$ is bounded and fine continuous.

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