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RECIPROCAL PROCESSES: THE STATIONARY GAUSSIAN CASE

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0. Introduction. Let $\{X_t, a < t < b\}$ be a stochastic process. Suppose that, for each $t \in (a, b)$, the σ-field generated by $\{X_s; a < s < t\}$ is conditionally independent, given X_t, of each event in the σ-field generated by $\{X_s: t < s < b\}$. Then $\{X_t, a < t < b\}$ is called a *Markov process*. Suppose instead that for each s, t in (a, b) with $s < t$ the following holds: each event in the σ-field generated by $\{X_r: s < r < t\}$ is conditionally independent, given X_s and X_t, of each event in the σ-field generated by $\{X_r: a < r < s\} \cup \{X_r: t < r < b\}$. We then call $\{X_t, a < t < b\}$ a *reciprocal process*. The use of the word "reciprocal" to describe this property is due to S. Bernstein [1]. In 1961 Slepian [7] noticed that the stationary Gaussian process with covariance function triangular on $[-1, 1]$ and zero outside $[-1, 1]$ has the reciprocal property on [0, 1], and exploited this property to compute explicitly the first passage time probability density for the restriction of the process to an interval of length 1. We address ourselves to the task of finding other real-valued stationary Gaussian processes having the reciprocal property on a finite or infinite interval. The natural approach to take seems that of Doob (see [2], pages 90–91 and 233–234). Exploiting the fact that in a Gaussian process conditioning is projection, Doob geometrizes the problem, and shows that a stationary Gaussian process is Markov if and only if its covariance function satisfies the functional equation for the exponential function. We geometrize our problem in a similar way, and succeed in showing that if a stationary Gaussian process is reciprocal then its covariance function satisfies a functional equation of a type satisfied by the cosine function. (I have L. A. Shepp to thank for the observation that the functional equation (11) is of this type, and for his reference to current literature on such functional equations.) The continuous functions which satisfy such functional equations on the whole real line were found by Cauchy. These were shown to be all the measurable solutions by Kacmarz [4]. We adapt the argument of Kacmarz to show that these functions ((15), (16), and (17) of our paper) are the only ones which satisfy such functional equations on an open interval. From these functions we select the covariances and obtain the following result. Suppose $\{X_t, 0 < t < T\}$ is a real-valued stationary Gaussian process with continuous covariance. It is reciprocal if and only if one of the following holds: (A) $\{X_t, 0 < t < T\}$ is Markovian, (B) $\{X_t, 0 < t < T\}$ is the restriction to $(0, T)$ of a sine wave of random phase and amplitude but of fixed period no less than $2T$, or (C) $\{X_t, 0 < t < T\}$ is any of a multitude of processes whose covariance function has a graph which is linear on $[0, T]$.

1. Main results. Suppose $\{X_t, a < t < b\}$ is a stochastic process on the open interval (a, b), where $-\infty \leqq a < b \leqq \infty$. Let $(\Omega, \mathscr{F}, P)$ be the underlying prob-

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