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DISTRIBUTIONS OF Z^{γ} AND Z^* FOR COMPLEX Z WITH RESULTS APPLIED TO THE COMPLEX NORMAL DISTRIBUTION[1]

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1. Introduction. Let $Z = Re^{i\Theta}$ be a complex random variable such that the density of (R, Θ) is given by

$$f(r, \theta) = \frac{|bc_4|\, c_2^{\,c_1}(1-a^2)^{\frac{1}{2}}}{2\pi m\Gamma(c_1)} r^{c_1c_4-1}\lambda^{c_1c_3-1}\exp(-c_2\lambda^{c_3}r^{c_4}) \tag{1.1}$$

where all parameters are real, $r > 0$, $m\pi/|b| < \theta < m\pi/|b|$, $\lambda = 1 - a\sin(b\theta + \alpha)$, $|a| < 1$, $b \neq 0$, $c_1 > 0$, $c_2 > 0$, $c_4 \neq 0$, m a natural number, and $0 \leqq \alpha < 2\pi$ and where $f(r, \theta)$ is zero otherwise. The generalized Mellin transform (GMT) (see [2]) will be used in order to obtain the distribution of Z^{γ} and Z^* when γ is a nonzero real number and Z^* is the complex conjugate of Z. The density function (1.1) is of special interest due to the importance of certain special cases of the family. In particular: (i) Weibull-uniform, (ii) chi-uniform, (iii) gamma-uniform, (iv) complex normal.

2. The distributions of Z^{γ} and Z^*.

THEOREM 2.1. *The* GMT *of the complex random variable* $Z = Re^{i\Theta}$, *with density of* (R, Θ) *given by* (1.1), *is*

$$h(s, t) = \frac{-(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4 + c_1)\sin(mt\pi/b)}{4\pi m c_2^{\,s/c_4}\Gamma(c_1)\Gamma(c_3 s/c_4 + 1)} \cdot \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} H_{jl}[(t/b)\sin^j\alpha + ij\sin^{j-1}\alpha\cos\alpha] \tag{2.1}$$

where

$$H_{jl} = \frac{(-1)^{mj}\Gamma(2l+j+c_3 s/c_4+1)\Gamma(j/2+t/2b)\Gamma(j/2-t/2b)a^{2l+j}}{2^{2l}\Gamma(j+1)\Gamma(j/2+t/2+l+1)\Gamma(j/2-t/2+l+1)} \tag{2.2}$$

and $t \neq 0, \pm b, \pm 2b, \cdots$, and $s/c_4 + c > 0$. [for any integer k, $h(s, kb)$ is evaluated by taking the limit of $h(s, t)$ as $t \to kb$.]

PROOF. By definition, the GMT of Z is

$$h(s, t) = \frac{|bc_4|\, c_2^{\,c_1}(1-a^2)^{\frac{1}{2}}}{2\pi m\Gamma(c_1)} \int_{-m\pi/|b|}^{m\pi/|b|}\int_0^{\infty} r^{s+c_1c_4-1}\lambda^{c_1c_3-1} \cdot [\exp(it\theta - c_2\lambda^{c_3}r^{c_4})]\, dr\, d\theta. \tag{2.3}$$

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