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ON THE FIRST TIME $|S_n| > cn^{\frac{1}{2}}$ [1]

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1. Introduction. Let $X_1, X_2, \cdots$ be an infinite sequence of independent, identically distributed (i.i.d.) random variables having a finite mean μ and a finite, positive variance σ^2 and consider the stopping time N defined by

$$N = \text{least } n \geqq 1 \text{ for which } |S_n| > cn^{\frac{1}{2}} \text{ or } +\infty \text{ if no such } n \text{ exists,} \tag{1.1}$$

where c is a positive constant and $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. Obviously, $N < \infty$ w.p. one (by the Strong Law of Large Numbers and the Law of the Iterated Logarithm), but if $\mu = 0$, then $E(N) < \infty$ if and only if $c^2 < \sigma^2$ ([1], [3]). Here we will consider the case $c^2 > \sigma^2$ and will investigate the rate at which $E(N)$ diverges to infinity as $\mu \to 0$. Our results assert the existence of positive constants b_1, b_2, γ_1, and γ_2 for which $0 < \gamma_1 < \gamma_2 < 1$ and

$$b_1 |\mu|^{-(1+\gamma_1)} \leqq E(N) \leqq b_2 |\mu|^{-(1+\gamma_2)} \tag{1.2}$$

for all sufficiently small values of μ. The constants b_1 and γ_1 depend only on c^2 and σ^2 and exist when $c^2 > 2\sigma^2$; the constants b_2 and γ_2 depend also on the distribution of $(X_i - \mu)/\sigma$ and require higher moments. Explicit values are given for all constants, and it is shown that γ_1 may be made arbitrarily close to one by taking c sufficiently large.

The left side of (1.2) is established in Section 2 and the right side in Section 3. An application to testing the sign of a bias is given in Section 4.

2. The lower bound. Throughout this section and the next we will assume the X's to be i.i.d. with mean μ and finite, positive variance σ^2. We begin with a variant on Wald's Lemma.

LEMMA 2.1. *Let* $0 < \alpha \leqq 1$ *and let* $\beta = 1 - \alpha$; *then*

$$E(N^{-\beta} S_N^2) \leqq 4c|\mu|(1+2\alpha)^{-1} E(N^{\frac{1}{2}+\alpha}) + \alpha^{-1}(\sigma^2 + \mu^2) E(N^\alpha).$$

PROOF. Without loss of generality, we may assume that $E(N^\alpha) < \infty$, in which case

$$n^{-\beta} \int_{N>n} S_n^2 \, dP \leqq c^2 n^\alpha P(N > n) \to 0 \tag{2.1}$$

as $n \to \infty$. Now for any $k \geqq 2$ we may write

$$\int_{N \leqq k} N^{-\beta} S_N^2 \, dP = \int_{N=1} S_1^2 \, dP + \sum_{n=2}^{k} \left[n^{-\beta} \int_{N>n-1} S_n^2 \, dP - n^{-\beta} \int_{N>n} S_n^2 \, dP \right]. \tag{2.2}$$

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