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# ON THE DEPARTURE FROM NORMALITY OF A CERTAIN CLASS OF MARTINGALES

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Let $\{X_n, \mathscr{F}_n, n = 0, 1, 2, \cdots\}$ be a martingale with $X_0 = 0$ a.s., $X_n = \sum_{i=1}^n Y_i$, $n \geqq 1$, and $\mathscr{F}_n$ the $\sigma$-field generated by $X_0, X_1, \cdots, X_n$. Write

$$\sigma_n^2 = E(Y_n^2 \mid \mathscr{F}_{n-1}), \qquad s_n^2 = \sum_{i=1}^n E\sigma_i^2,$$

and suppose that there is a constant $\delta$, with $0 < \delta \leqq 1$, such that $E|Y_n|^{2+2\delta} < \infty$, $n = 1, 2, \cdots$. It is the object of this paper to establish the following theorem on departure from normality.

THEOREM. *There exist finite constants $K_1$, $K_2$ depending only on $\delta$, such that*

$$\begin{aligned} &\sup_x |P(X_n \leqq s_n x) - \Phi(x)| \\ &\leqq K_1\{s_n^{-2-2\delta}(\textstyle\sum_{i=1}^n E|Y_i|^{2+2\delta} + E|(\sum_{i=1}^n \sigma_i^2) - s_n^2|^{1+\delta})\}^{1/(3+2\delta)} \\ &\leqq K_2\{s_n^{-2-2\delta}(\textstyle\sum_{i=1}^n E|Y_i|^{2+2\delta} + E|(\sum_{i=1}^n Y_i^2) - s_n^2|^{1+\delta})\}^{1/(3+2\delta)}, \end{aligned} \tag{1}$$

*where*

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{1}{2}u^2}\, du.$$

*Thus, if*

$$\lim_{n\to\infty} s_n^{-2-2\delta} \sum_{i=1}^n E|Y_i|^{2+2\delta} = 0, \tag{2}$$

*and*

$$\lim_{n\to\infty} E|(s_n^{-2} \sum_{i=1}^n Y_i^2) - 1|^{1+\delta} = 0 \tag{3}$$

*or more generally*, (2) *and*

$$\lim_{n\to\infty} E|(s_n^{-2} \sum_{i=1}^n \sigma_i^2) - 1|^{1+\delta} = 0, \tag{4}$$

*then* $\lim_{n\to\infty} P(X_n \leqq s_n x) = \Phi(x)$, *and a bound on the rate of convergence is given by* (1).

The interesting feature of this result is the bound on departure from normality. More general central limit results for martingales are known, under conditions related to (2), (3) and (4) (see for example Brown [1]), but rates of convergence are not available.

We note that if $Y_1, Y_2, \cdots$ are independent, or more generally if $\sigma_1^2, \sigma_2^2, \cdots$ are constants a.s., then $s_n^{-2}\sum_{i=1}^n \sigma_i^2 = 1$ a.s., (4) is trivially true, and the first bound in (1) assumes a simplified form, since its second term vanishes. The utility of the second bound in (1) is that it does not depend on the conditioning by the sequence of $\sigma$-fields $\{\mathscr{F}_n\}$.

The proof of the theorem is based on a martingale form of the Skorokhod representation theorem which we state as a lemma in the interests of clarity.

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