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REVERSE SUBMARTINGALE AND SOME FUNCTIONS OF ORDER STATISTICS

BY B. B. BHATTACHARYYA

North Carolina State University

1. Introduction. Let $\{X_n, n \geqq 1\}$ be an exchangeable sequence of random variables and $Y_{1,n} \leqq Y_{2,n} \leqq \cdots \leqq Y_{n,n}$ be the order statistics based on $X_1, X_2, \cdots, X_n$. The object of this note is to show that $(Y_{n,n} - Y_{1,n})/\binom{n}{2}$ forms a reverse submartingale sequence of random variables, if $E(X_i) < \infty$. Moreover, if the X_i's are nonnegative random variables then $Y_{n,n}/n$ also forms a reverse submartingale sequence. Some moment properties of these statistics follow from these observations. We have also shown that an upper bound of $E(Y_{n,n} - Y_{1,n})$ is the expected value of range of n observations from a sequence of independent and identically distributed random variables having the same marginal distribution as that of X_i.

2. Some inequalities. Consider a set of $n+m$ finite real numbers $x_1, x_2, \cdots, x_{n+m}$ with $y_1 \leqq y_2 \leqq \cdots \leqq y_{n+m}$ as the corresponding ordered set. Let $(x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})$, $i = 1, 2, \cdots, \binom{n+m}{n}$, be the all possible subsets of n tuples that can be formed from the $n+m$ x's and $\{y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}\}$ denote the corresponding ordered set. The range of $(x_1, x_2, \cdots, x_{n+m})$ is $R_{n+m} = y_{n+m} - y_1 = \max_{1 \leqq h \neq k \leqq n+m} |x_h - x_k|$. The range of the ith subset of n x's is indicated by $R_n^{(i)}$.

LEMMA 1.

$$\binom{n+m}{n}\binom{n}{2} R_{m+n} \leqq \binom{n+m}{2} \sum_{i=1}^{\binom{n+m}{n}} R_n^{(i)}. \tag{2.1}$$

PROOF. Notice that there are $\binom{m+n-2}{n-2}$ subsets in which $R_n^{(i)}$ is greater or equal to $|x_h - x_k|$. Hence

$$\binom{n+m}{2} \sum_{i=1}^{\binom{n+m}{n}} R_n^{(i)} \geqq \binom{n+m}{2}\binom{m+n-2}{n-2} |x_h - x_k| = \binom{n+m}{n}\binom{n}{2} |x_h - x_k|. \tag{2.2}$$

The lemma follows by taking the maximum on both sides of (2.2).

LEMMA 2. *If the x_i's are nonnegative, then*

$$\binom{m+n}{n} n y_{n+m} \leqq (n+m) \sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)}. \tag{2.3}$$

PROOF. We know that $y_n^{(i)} \geqq y_j$ for all $j \leqq n$. Hence

$$(n+m) \sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)} \geqq \binom{n+m}{n} n y_j \qquad \text{for} \quad j \leqq n \quad \text{and} \quad m \geqq 0. \tag{2.4}$$

For $j > n$, there are $\binom{n+m}{n} - \binom{j-1}{n}$ subsets where $y_n^{(i)} \geqq y_j$. Hence

$$\begin{aligned} (n+m) \sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)} &\geqq (n+m)\left[\binom{n+m}{n} - \binom{j-1}{n}\right] y_j \\ &\geqq (n+m)\left[\binom{n+m}{n} - \binom{n+m-1}{n}\right] y_j \\ &= \binom{n+m}{n} n y_j \qquad \text{for} \quad j > n. \end{aligned} \tag{2.5}$$

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