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ON THE RELATION BETWEEN BAHADUR EFFICIENCY AND POWER

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1. Definitions and introduction. For a set of probability measures $\{P_\theta\}$, $\theta \in \Omega$, defined on an abstract sample space S, let H be the hypothesis: $\theta \in \Omega_0 \subset \Omega$, and let $\{T_n\}$ be a sequence of test statistics defined on S. Following Bahadur [1], $\{T_n\}$ is called a *standard sequence* if (I) for each $\theta \in \Omega_0$ there exists a continuous probability distribution function $F(x)$ such that $\lim_{n\to\infty} P_\theta\{T_n < x\} = F(x)$ for each x, (II) there is a positive constant a such that $\log[1-F(x)] = -\frac{1}{2}ax^2[1+o(1)]$ as $x \to \infty$, and (III) there is a function $b(\theta)$, $0 < b(\theta) < \infty$, on $\Omega - \Omega_0$ such that $\lim_{n\to\infty} P_\theta\{|T_n/n^{\frac{1}{2}} - b(\theta)| > \varepsilon\} = 0$ for each $\varepsilon > 0$ and for each $\theta \in \Omega - \Omega_0$. The function $c(\theta)$ on Ω defined by $c(\theta) = 0$ for $\theta \in \Omega_0$ and $c(\theta) = ab^2(\theta)$ for $\theta \in \Omega - \Omega_0$ is called the *slope* of $\{T_n\}$ when θ obtains. If $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ are two standard sequences with slopes $c_1(\theta)$ and $c_2(\theta)$, respectively, then the ratio $\phi_{12}(\theta) = c_1(\theta)/c_2(\theta)$ is called the asymptotic efficiency of $\{T_n^{(1)}\}$ relative to $\{T_n^{(2)}\}$. For $0 < \alpha < 1$ and $\theta \in \Omega$, let $\beta_n^{(i)}(\alpha \mid \theta) = P_\theta\{F^{(i)}(T_n^{(i)}) < 1-\alpha\}$, $i = 1, 2$, and let $\delta_n(1, 2 \mid \theta) = \sup_\alpha [\beta_n^{(2)}(\alpha \mid \theta) - \beta_n^{(1)}(\alpha \mid \theta)]$. Then $\{T_n^{(2)}\}$ is said to *dominate* $\{T_n^{(1)}\}$ at θ if $\lim_{n\to\infty} \delta_n(1, 2 \mid \theta) = 0$.

The function ϕ_{12} has many important and interesting properties (cf. [1], [2], [3]) some of which concern the power of the test statistics. Among other things, R. R. Bahadur [1] shows (i) if $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ at θ, then $\phi_{12}(\theta) \leqq 1$ and (ii) if $\phi_{12}(\theta) < 1$, then $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ at θ. The assertion is then made that from these results it follows that $\phi_{12} < 1$ if and only if $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$. However, this is not the case. If $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$, one can conclude from (i) and (ii) only that $\phi_{12} \leqq 1$. To illustrate that $\phi_{12} \leqq 1$ is the correct conclusion rather than $\phi_{12} < 1$, the authors give an example of two standard sequences $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ for which $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$, and for which $\phi_{12} = 1$.

2. Example. Let $s = (x_1, x_2, \cdots)$ where the x_i are independent and normally distributed on the real line with $E(x_i) = \theta$, $\operatorname{Var}(x_i) = 1$. Let Ω consist of two points 0 and μ, where $\mu > 0$, and let H be the hypothesis that $\theta = 0$. For each $n = 1, 2, \cdots$ let $T_n^{(2)} = \sum_{i=1}^n x_i/n^{\frac{1}{2}}$. It is well known and readily verified that $\{T_n^{(2)}\}$ is a standard sequence and its slope $c_2(\mu) = \mu^2$. Let $k_1 \leqq k_2 \leqq \cdots$ be a sequence of positive integers such that

$$k_n \leqq n \qquad \text{for each } n, \tag{1}$$

and

$$k_n/n \to 1 \qquad \text{as } n \to \infty, \tag{2}$$

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