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NOTES

HARMONIC FUNCTIONS AND HITTING DISTRIBUTIONS FOR MARKOV PROCESSES[1]

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1. Introduction. In one dimension it is well known that a homeomorphism φ takes a continuous standard Markov process (Y, T_E) into a process having Brownian hitting distributions if and only if φ is a harmonic function for (Y, T_E). It is shown that in the complex plane the homeomorphism φ takes a continuous standard Markov process (Y, T_E) into a process having Brownian hitting distributions if and only if the functions φ^k, $k \geqq 0$, of a complex variable are complex harmonic for (Y, T_E) (i.e., real and imaginary parts are harmonic). Specifically, if (Y, T_E) has Brownian hitting distributions then $(\varphi(Y), T_{\varphi(E)})$ has Brownian hitting distributions if and only if φ or φ conjugate is analytic in E. It seems that these arguments may be extended to higher dimensions and the spherical harmonic functions; they are presented here in this framework because of their simplicity.

2. Notation. We shall call $(Y, T_E) = (Y_t, T_E, F_t, P_x)$ a process if, in the notation of Dynkin [1], it is a continuous standard Markov process with sample paths in a region E of the complex plane, and its harmonic functions are continuous. $(T_{\partial E}(\omega))$ is the infimum of $t > 0$ such that Y_t is in ∂E, if that infimum exists, and $+\infty$ otherwise. The function φ is said to be harmonic for $(Y, T_{\partial E})$ if $\varphi(x) = \mathscr{E}_x \varphi(Y_{T_{\partial U}})$ for x in E, U open, $\bar{U} \subset E$.) It follows that harmonic functions satisfy a maximum modulus theorem (thus the harmonic functions are continuous if the excessive functions are upper semi-continuous); hence, it is well known (c.f. [1]) that the following are equivalent for any real function φ:

(A) If T is a stopping time, $T < T_{\partial E}$ a.s. then $\varphi(x) = \mathscr{E}_x \varphi(Y_T)$.

(B) Given x, there exists a neighborhood $D \subset E$ of x such that $\varphi(x) = \mathscr{E}_x \varphi(Y_{T_{\partial N}})$ for all neighborhoods N of x, $N \subset D$.

Accordingly, we will take either of these to be our definition of harmonic, and we say a function is complex harmonic if its real and imaginary parts are harmonic.

3. Theorems. Let $(X, \tilde{T}_{\partial D})$ and $(Y, T_{\partial D})$ be two processes. (Stopping times for X will be denoted by $\tilde{T}$.)

We say that $(g, B) \in \mathscr{H}_X$ (alternatively $(g, B) \in \mathscr{H}_Y$) when B is a bounded region, $\bar{B} \subset D$, and g is a real harmonic function for the process $(X, \tilde{T}_{\partial B})$ (alternatively $(Y, T_{\partial B})$).

In the following theorems and discussions the process $(X, \tilde{T}_{\partial D})$ will always be

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