The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 2113–2122

A CHARACTERIZATION BASED ON THE ABSOLUTE DIFFERENCE OF TWO I.I.D. RANDOM VARIABLES

BY PREM S. PURI[1] AND HERMAN RUBIN[2]

Purdue University

1. Introduction. Let X_1 and X_2 be two independent and identically distributed (i.i.d.) random variables whose common distribution is the same as that of a random variable X. The problem considered here is to characterize all possible distributions of X which satisfy the following property H:

(1) $\quad H$: The distribution of $|X_1 - X_2|$ and X are identical.

For instance, it is easy to verify that the discrete distribution with $P(X = 0) = P(X = a) = \frac{1}{2}$ for some positive constant a, and the exponential distribution with probability density function (pdf) f where $f(x) = \theta \exp(-\theta x)$, for $x \geqq 0$, and $f(x) = 0$ elsewhere, with $\theta > 0$, both satisfy the property H. The reader may find a different characterization based on $|X_1 - X_2|$ in Puri [6]. Basu [1], Ferguson ([4], [5]) and Crawford [2] have considered a different problem where they characterize distributions with the property that $\min(X_1, X_2)$ is independent of $X_1 - X_2$. Their methods naturally depend very heavily upon such an independence, which of course is lacking in the present case.

Let F denote the distribution function (df) of X. It can be easily shown that if X satisfies H, the distribution of X can either be only discrete or absolutely continuous or singular and no mixture is possible. Thus one needs to consider these three possibilities separately. For the case when X is discrete let A denote the set of possible discrete nonnegative values that X takes. More specifically, let

$$p_y = P(X = y), \qquad y \in A; \quad \text{with} \quad \textstyle\sum_{y \in A} p_y = 1.$$

It is clear that if there exists a $y \geqq 0$ with $p_y > 0$, then in particular A contains zero with $p_0 > 0$. Furthermore, from the property H, the following relations follow easily:

$$p_0 = \textstyle\sum_{x \geqq 0} p_x^{\,2}, \tag{2}$$

$$p_y = 2 \textstyle\sum_{x \geqq 0} p_x \, p_{x+y}; \qquad y > 0. \tag{3}$$

Similar relations are satisfied by the pdf f if X satisfying H is absolutely continuous.

In Section 2, we show that under H, X has a moment generating function (mgf) and hence all its moments are finite. Also in Theorem 1, we consider the case

Received January 26, 1970; revised June 3, 1970.

[1] This investigation was supported in part by research grant GM-10525 from NIH, Public Health Service, at the University of California, Berkeley.

[2] This research was partly supported by the Office of Naval Research Contract N00014-67-A-0226-0008, project NR042-216. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.