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DESIGNS FOR REGRESSION PROBLEMS WITH CORRELATED ERRORS III

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1. Introduction. Consider the linear regression model in which one may observe a stochastic process Y having the form

$$Y(t) = \sum_{j=1}^{J} \beta_j f_j(t) + Z(t), \qquad t \in [0, 1]. \tag{1.1}$$

Here the β_j are taken as unknown constants, the f_j as known functions and Z is assumed to have mean function zero and known covariance kernel R. Let T be a subset of [0, 1] and let $\hat{\beta}_T$ denote the best linear estimate (if it exists) of $\beta = (\beta_1, \cdots, \beta_J)'$ based on observing $\{Y(t), t \in T\}$. When the covariance matrix of $\hat{\beta}_T$ is nonsingular it will be denoted by A_T^{-1}; when $T = [0, 1]$ we will use the notation A^{-1}.

In an earlier paper [1], we treated the special case $J = 1$ of (1.1). The problem posed was that of finding a member T_n in the class $\mathscr{D}_n = \{T \mid T = \{t_0, t_1, \cdots, t_n\}, 0 = t_0 < t_1 < \cdots < t_n = 1\}$ of all $n+1$ point "designs" for which $A_{T_n}^{-1} = \inf_{T \in \mathscr{D}_n} A_T^{-1}$. We assumed there that $f_1 = f$ had the form

$$f(t) = \int_0^1 R(s, t)\varphi(s)\, ds \tag{1.2}$$

for some continuous function φ and that R satisfied assumptions slightly weaker than those labelled A, B and C in Section 2 below (see also the Remark at the end of Section 2). It was then shown that

$$\inf_{T \in \mathscr{D}_n} A_T^{-1} - A^{-1} = \frac{c^3(\varphi)}{12n^2A^2} + o(1) \tag{1.3}$$

$$A_{T_n^*}^{-1} - A^{-1} = \frac{c^3(\varphi)}{12n^2A^2} + o(1) \tag{1.4}$$

where T_n^* is a set of n-tiles of the probability distribution function with density $c^{-1}(\varphi)\varphi^{\frac{2}{3}}$. Thus our approximate solution to the design problem in $\mathscr{D}_n$ is T_n^*. We say when (1.3) and (1.4) are satisfied that sampling according to $\varphi^{\frac{2}{3}}$ is asymptotically optimum.

In a second paper [2], the full model (1.1) was discussed. There, for a variety of criteria ψ which would measure the size of A_T^{-1} (e.g. the generalized variance), we sought T_n in $\mathscr{D}_n$ for which $\psi(A_{T_n}^{-1}) = \inf_{T \in \mathscr{D}_n} \psi(A_T^{-1})$. It was assumed that each f_j had the form (1.2) with associated φ_j and that R was subject to the same restrictions as in [1]. Our results then had the following character: given a criterion

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