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CONVERGENCE OF SUMS OF RANDOM VARIABLES CONDITIONED ON A FUTURE CHANGE OF SIGN[1]

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1. Introduction and notation. Let $\{\xi_k\}$ be a sequence of independent and identically distributed random variables which have a non-lattice distribution and which are in the domain of attraction of a non-degenerate stable law of index $\alpha(0 < \alpha \leqq 2)$. Let S_n denote the partial sum $\xi_1 + \cdots + \xi_n$. It is the purpose of this paper to study the limiting behavior of $\{S_k, k \leqq n\}$ as $n \to \infty$ under the condition that $S_n S_{n+1} < 0$. It turns out that the results depend on whether $\alpha < 1$ or $\alpha \geqq 1$ as is so often the case in the study of stable laws and their domains of attraction.

We will assume for simplicity that no centering constants are needed for the convergence of the normalized partial sums. So, there are constants $b_n > 0$ so that S_n/b_n converges in distribution to an appropriate stable law, which we will call $X(1)$. For each n, define a stochastic process $X_n(t)$ by

$$\begin{aligned} X_n(t) &= S_{[(n+1)t]}/b_n \qquad \text{for} \quad 0 \leqq t < 1 \\ &= S_n/b_n \qquad\qquad \text{for} \quad t = 1. \end{aligned}$$

These processes are regarded as random elements of Skorokhod's space $D[0, 1]$ (see Chapter 3 of [1]). It then follows from a theorem of Skorokhod that $X_n(t)$ converges weakly in $D[0, 1]$ to a stable process $X(t)$ whose one-dimensional distributions are the same as those of $t^{1/\alpha}X(1)$ (see Theorem 1 of [5]).

The processes of interest here are $(X_n(t) \mid S_n S_{n+1} < 0)$, which again are regarded as random elements of $D[0, 1]$. In order to insure that the conditioning event $(S_n S_{n+1} < 0)$ has positive probability for each n, we will assume that the stable law $X(1)$ is not one-sided. The main result of this paper is that the processes $(X_n \mid S_n S_{n+1} < 0)$ converge weakly to a limiting process which concentrates at the origin at time one if $1 \leqq \alpha \leqq 2$ and which concentrates on the whole line at time one if $\alpha < 1$.

In two earlier papers, the author studied the limiting behavior of processes of the form $(X_n(t) \mid X_n(1) \in E^n)$ where E^n is a sequence of Borel subsets of the real line ([5] and [6]). Theorem 1 of this paper may be regarded as an application of the results of [6].

2. The case $1 \leqq \alpha \leqq 2$. We will show that in this case $(X_n(t) \mid S_n S_{n+1} < 0)$ converges to the process which is obtained by "tying" $X(t)$ down to the origin at time one. For $\alpha = 2$, this process is the ordinary Brownian Bridge.

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