The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 1970–1977

A CHARACTERISTIC PROPERTY OF THE MULTIVARIATE NORMAL DENSITY FUNCTION AND SOME OF ITS APPLICATIONS

BY G. P. PATIL AND M. T. BOSWELL

The Pennsylvania State University

0. Introduction. Price [9] proved by using Dirac δ-functions the following two theorems in a different form.

THEOREM 0.1. *If $X_1, \cdots, X_n$ have an n-variate normal distribution with unit variances and if $g_1, \cdots, g_n$ are functions respectively of $X_1, \cdots, X_n$ admitting Laplace transforms, then*

$$\frac{\partial}{\partial \sigma_{jk}} E\left[\prod_{m=1}^{n} g_m(X_m)\right] = E\left[\frac{\partial^2}{\partial X_j \partial X_k} \prod_{m=1}^{n} g_m(X_m)\right] \tag{0.1}$$

where σ_{jk} is the correlation coefficient of X_j and X_k.

THEOREM 0.2. *If* (0.1) *holds for arbitrary functions $g_1, \cdots, g_n$, then $X_1, \cdots, X_n$ have an n-variate normal distribution.*

For $n = 2$, McMahon [8] showed that

$$\frac{\partial}{\partial \sigma_{12}} E[g(X_1, X_2)] = E\left[\frac{\partial^2}{\partial X_1 \partial X_2} g(X_1, X_2)\right] \tag{0.2}$$

where g is a function of both X_1 and X_2, admitting 2-dimensional Laplace transform. Papoulis [9] generalized (0.2) to the case where $|g(x_1, x_2)| \leqq a \exp(x_1^{\alpha} + x_2^{\alpha})$, $a > 0$, $\alpha < 2$, is required instead of requiring the existence of the Laplace transform. Brown [1] pointed out that the result and method of Papoulis generalize directly to higher dimensions, providing certain generalizations of Price's theorems.

A major motivation in the above-mentioned work lies in obtaining $E[g(X_1, \cdots, X_n)]$ for a multivariate normal distribution by solving a partial differential equation of the kind in (0.2). This method has appeared in books and papers and has sometimes been used where the theoretical justification is lacking. Pawula in [7] gives an alternate approach where the covariance matrix is modified so that each non-diagonal element is multiplied by α, and differentiation is with respect to α.

Section 1 presents a property of the multivariate normal density function. This fundamental theorem generalizes the work of Plackett [8]. Section 2 provides a generalization of Price's theorem with a rigorous proof and presents some of its applications as corollaries. Corollary 2.2 uses Price's theorem as a theoretical tool to solve a problem that arose in an estimation problem [2]. Characterizations of the multivariate normal are studied in Section 3, and in Section 4. The relation of moments to independence, analogous to "decorrelations" discussed in Linnik [3], is studied.

Received December 2, 1968.