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LINEAR RANK STATISTICS UNDER ALTERNATIVES INDEXED BY A VECTOR PARAMETER[1]

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1. Introduction. Since the publication of Hájek's papers [3] and [4] on the asymptotic distributions of linear rank statistics under the null hypothesis of randomness and under contiguous location-shift alternatives, several other fruitful applications of the same basic techniques have been made. Some of these later applications may be found in the papers by Matthes and Truax [7] and by Adichie [1], as well as in the book [5] by Hájek and Šidák. With the exception of [1], the contiguous alternatives examined in these references have depended upon a one-dimensional parameter. Adichie considered a two-dimensional parameter indexing simple regression alternatives, but by making the parameter components linearly dependent in the asymptotics, essentially reduced the problem.

This paper presents a general treatment of the asymptotic distributions of linear rank statistics under contiguous alternatives indexed by a q-dimensional parameter. The approach adopted follows that of Hájek and Šidák [5]. It is assumed that under the null hypothesis, the observations are independent and identically distributed random variables. The linear rank statistics considered are of the forms

$$S_c = \sum_{i=1}^{N} \mathbf{c}_i' \mathbf{a}_N(R_{Ni}), \quad \text{and} \tag{1.1}$$

$$S_c^{+} = \sum_{i=1}^{N} \mathbf{c}_i' \mathbf{a}_N(R_{Ni}^{+}) \operatorname{sign}(X_i), \tag{1.2}$$

where the p-dimensional column vectors $\mathbf{c}_1, \cdots, \mathbf{c}_N$ and $\mathbf{a}_N(1), \cdots, \mathbf{a}_N(N)$ are, respectively, constants and values of a score function $\mathbf{a}_N(\cdot)$. Such statistics arise naturally in the present context and also in the study of locally most powerful rank tests. The results of this paper may be used to derive asymptotic distribution theory under vector parameter alternatives for the Kolmogorov–Smirnov, Cramér–von Mises and Rényi statistics as well as for various simple quadratic rank statistics. Section 4 illustrates.

2. Limiting distributions under the null hypothesis. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with common density f, defined on R^1. Let R_{Ni} denote the rank of X_i among $X_1, \cdots, X_N$. The first situation envisaged in this section is that of sequences of sample sizes $\{N_\nu\}$, of p-dimensional vector constants $\{(\mathbf{c}_{\nu 1}, \cdots, \mathbf{c}_{\nu N_\nu})\}$, of linear rank statistics $\{S_{c\nu}\}$, defined as in (1.1), and of null hypotheses $\{H_\nu\}$. Under H_ν, the joint density of $(X_1, \cdots, X_{N_\nu})$ is assumed to be

$$p_\nu(\mathbf{x}) = \prod_{i=1}^{N_\nu} f(x_i). \tag{2.1}$$

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