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ON UNIVERSAL MEASURABILITY AND PERFECT PROBABILITY

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A subset E of the real numbers R is an element of the set $\mathscr{U}$ of universally measurable subsets of R if, and only if, $\mu^*(E) = \mu_*(E)$ for each probability measure μ on the Borel subsets $\mathscr{B}$ of R. A subset E of R is an element of the sigma ideal $\mathscr{N}$ of universal null sets if, and only if, $\mu^*(E) = 0$ whenever μ is a nonatomic probability measure on $\mathscr{B}$.

The purpose of this note is to recount some properties of the sigma algebra $\mathscr{U}$ and its sigma ideal $\mathscr{N}$.

When dealing with a finite, nonnegative measure μ on a sigma algebra $\mathscr{S}$ of subsets of a set X it suffices, for our purposes, to normalize and, hence, suppose that μ is an element of the set $\mathscr{P}(\mathscr{S})$ of probability measures on $\mathscr{S}$. An element μ of $\mathscr{P}(\mathscr{S})$ is said to be perfect if for each $\mathscr{S}$-measurable function f, there exists $B \in \mathscr{B}$ such that $B \subset f(X)$ and $\mu(f^{-1}(B)) = 1$.

If A is a subset of R, then $\mathscr{B}_A$ will denote the sigma algebra of Borel subsets of A.

D. Blackwell [1] used

(1) If A is an analytic subset of R and f is a $\mathscr{B}_A$-measurable function, then $f(A)$ is an analytic set, and

(2) The sigma algebra of subsets of R generated by the analytic sets in a subset of $\mathscr{U}$, to show

(3) If A is an analytic subset of the interval $I = [0, 1]$ and $\mu \in \mathscr{P}(\mathscr{B}_A)$, then μ is perfect.

Then he asked whether there be subsets A of I, other than analytic sets, with the property that every $\mu \in \mathscr{P}(\mathscr{B}_A)$ is perfect.

V. V. Sazonov ([6] Lemma 3) answered Blackwell's question by showing that

(4) The necessary and sufficient condition in order that every $\mu \in \mathscr{P}(\mathscr{B}_A)$ be perfect is that $A \in \mathscr{U}$.

Meanwhile, G. Kallianpur introduced the notion of a D-space (i.e., $(A, \mathscr{B}_A)$ is a D-space if, and only if, $f(A) \in \mathscr{U}$ for every $\mathscr{B}_A$-measurable function f) in [3]. He showed that

(5) The necessary and sufficient condition in order that $(A, \mathscr{B}_A)$ be a D-space is that for every separable subsigma algebra $\mathscr{A}$ of $\mathscr{B}_A$, every $\mu \in \mathscr{P}(\mathscr{A})$ is perfect.

Unaware of Kallianpur's paper, Sazonov ([6] Theorem 9) gave a proof of the sufficiency of Kallianpur's result and a proof of the necessity of a stronger result:

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