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CONVERGENCE OF CONDITIONAL EXPECTATIONS

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0. Introduction. Let $(X, \mathscr{F})$ be a measurable space and $P_n|\mathscr{F}$, $n = 0, 1, 2, \cdots$, a family of probability measures such that $(P_n)_{n\in\mathbb{N}}$ converges to P_0 in some appropriate sense. Let $\mathscr{F}_0 \subset \mathscr{F}$ be an arbitrary sub-σ-field. For $f \in \bigcap_{n=0}^{\infty} \mathscr{L}_1(X, \mathscr{F}, P_n)$ let $p_n(f, \cdot)$ denote a conditional expectation of f relative to P_n, given $\mathscr{F}_0$. Our problem is to give sufficient conditions under which $(p_n(f, \cdot))_{n\in\mathbb{N}}$ converges to $p_0(f, \cdot)$ in some sense.

1. The results. Though the convergence of conditional expectations has been treated by a number of authors (see references) and various sufficient conditions have been given, the following rather natural condition seems to have been overlooked.

Let $\mu|\mathscr{F}$ be a σ-finite measure dominating $P_n|\mathscr{F}$ for all $n = 0, 1, 2, \cdots$. Let h_n be a density of $P_n|\mathscr{F}$ with respect to $\mu|\mathscr{F}$ and h_{0n} a density of $P_n|\mathscr{F}_0$ with respect to $\mu|\mathscr{F}_0$.

THEOREM 1. *Assume that*

(i) $(h_n)_{n\in\mathbb{N}}$ *converges to* h_0 μ-a.e.

(ii) $(h_{0n})_{n\in\mathbb{N}}$ *converges to* h_{00} μ-a.e.

Then

(1) *For abitrary versions of the conditional expectations*: $(p_n(f, \cdot))_{n\in\mathbb{N}}$ *converges to* $p_0(f, \cdot)$ P_0-a.e. *for any* $\mathscr{F}$*-measurable, bounded function* f.

(2) *If a regular conditional probability relative to* μ, *given* $\mathscr{F}_0$, *exists, then there exists a regular conditional probability* $p_n^*|\mathscr{F}\times X$ *relative to* P_n, *given* $\mathscr{F}_0$, *such that* $(\sup_{A\in\mathscr{F}} |p_n^*(A, \cdot) - p_0^*(A, \cdot)|)_{n\in\mathbb{N}}$ *converges to* 0 P_0-a.e.

(3) *If, in addition,* $\mathscr{F}$ *is countably generated, the uniform convergence asserted in* (2) *holds for all regular conditional probabilities* $p_n|\mathscr{F}\times X$ *relative to* P_n, *given* $\mathscr{F}_0$.

We remark that the conditions on the densities depend neither on the particular dominating measure μ nor on the particular versions chosen. Hence w.l.g. $\mu(X) = 1$.

PROOF. (i) Let $B_n = \{x \in X: h_{0n}(x) > 0\}$ and $q_n(x) = ((h_n(x)/h_{0n}(x))1_{B_n}(x) + 1_{\bar{B}_n}(x)$, $n = 0, 1, 2, \cdots$, (where $\bar{B}_n$ denotes the complementary set of B_n in X). Since $P_n(\bar{B}_n) = 0$, we have $h_n(x) = 0$ for μ-a.a. $x \in \bar{B}_n$ and therefore $h_{0n}q_n = h_n$ μ-a.e. For $g \in \mathscr{L}_1(X, \mathscr{F}, \mu)$ let $\mu(g, \cdot) \in \mathscr{L}_1(X, \mathscr{F}_0, \mu)$ denote a conditional expectation of g relative to μ, given $\mathscr{F}_0$. As $\mu(h_n, \cdot)$ is a density of $P_n|\mathscr{F}_0$ with respect to $\mu|\mathscr{F}_0$, we have $h_{0n} = \mu(h_n, \cdot)$ μ-a.e. and therefore $\mu(q_n, \cdot) = 1$ μ-a.e. This implies in particular that $\mu(q_n) = 1$ so that all functions $q_n 1_B$ are μ-integrable. (Here and in the following, $\mu(g)$ means $\int g(\xi)\,\mu(d\xi)$.)

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