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ON MEASURABLE GAMBLING PROBLEMS[1]

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1. Introduction. Let Γ be a measurable gambling house defined on a Borel set of fortunes F. (Precise definitions are given later.) Starting with fortune f, a gambler chooses a strategy σ available to him. The strategy σ induces a probability measure on the product space $H = F \times F \times \cdots$ of histories of fortunes and the gambler is paid $\int g\, d\sigma$, the expectation of g under σ, where g is some utility function on H. Let $M_g(f)$ be the sup $\int g\, d\sigma$ taken over all measurable strategies "essentially" available at f and let $\Gamma_g(f)$ be the same supremum taken over all strategies σ available at f. The function M_g is well-defined when g is a bounded, Borel measurable function and it is shown below that, in this case, M_g is universally measurable. The function Γ_g is well-defined if g is bounded and finitary. If g is bounded, finitary, and Borel, then both functions are well-defined and seen to be equal. Thus a gambler can do just as well when restricted to measurable strategies for these problems.

These results seem to contain most of the known results on the measurability of the return function and the adequacy of measurable strategies but the problem which motivated this research nevertheless remains open. That is, do good measurable strategies exist for measurable problems with a measurable utility function of the type studied by Dubins and Savage? (If so, the return function is universally measurable.)

Some progress is made on this question. Let u be a bounded function on F and σ a strategy. Then $u(\sigma)$ is defined to be $\limsup_{t\to\infty} \int u(f_t)\, d\sigma$, where the lim sup is over all stop rules t. It is shown below that for u and σ measurable, it is equivalent to take the lim sup over all measurable stop rules and also $u(\sigma) = \int u^*\, d\sigma$ where u^* is a bounded, measurable function on H. By the result previously mentioned, $M_u{}^*$ is universally measurable. The question remaining is whether $M_u{}^*$ is the optimal return function V studied by Dubins and Savage.

For expository reasons, the results for $u(\sigma)$ are presented first. However, the reader who wishes may skim Section 2 and skip to Section 5 and Section 6 for the results outlined in the first paragraph.

2. Measurable strategies. Let F be a set and let G be the set of all gambles on F. That is, G is the set of all finitely additive probability measures defined on all subsets of F. A *strategy* σ is a sequence $\sigma_0, \sigma_1, \cdots$ where $\sigma_0 \in G$ and, for $n \geqq 1$, σ_n maps $F \times \cdots \times F$ (n-factors) into G. Let H be the countably infinite product $F \times F \times \cdots$ and let g be a bounded, finitary function on H. Then $\int g\, d\sigma$ was defined in [2].

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