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CONSTANT COEFFICIENT LINEAR DIFFERENTIAL EQUATIONS DRIVEN BY WHITE NOISE

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1. Introduction. Consider the process w_t defined by the linear stochastic differential equation

$$\dot{w}_t = Aw_t + B\dot{\beta}_t,\ w_0 = c, \tag{1}$$

where $A \in R^{n \times n}$, $B \in R^{n \times p}$ ($R^{k \times l}$ = real $k \times l$ matrices, $R^n = R^{n \times 1}$) and β_t is the standard p-dimensional Brownian motion. The purpose of this note is to analyze the trajectories of this process, to find conditions for transience of w_t in terms of the eigenvalues of (a matrix related to) A, and to give the general form of the invariant densities for the process.

These results can be given in abbreviated form because of the groundwork laid by Dym [1] who has considered all of the above problems for the special case where w_t is a solution of

$$(D^n - a_1 D^{n-1} - \cdots - a_n)w_t = \dot{\beta}_t,\ w_0^{(k-1)} = c_k, \qquad k = 1, \cdots, n, \tag{2}$$

and β_t is one-dimensional Brownian motion.

Zakai and Snyders [4] give three equivalent necessary and sufficient conditions for the existence of a stationary probability measure for solutions of (1); two of these conditions are implicit in our final theorem.

2. The trajectories of w_t remain on an m-flat. The formal solution to (1) is given by

$$w_t = \int_0^t e^{(t-s)A}\, B\, d\beta_s + e^{tA} c \tag{3}$$

and is known to be a diffusion (see, e.g., Dynkin [2]). From the properties of stochastic integrals and Brownian motion it follows that w_t is Gaussian with mean

$$E^c(w_t) = e^{tA} c \tag{4}$$

and covariance

$$R_t = E^c((w_t - e^{tA}c)(w_t - e^{tA}c)^*) = \int_0^t e^{sA} BB^* e^{sA^*}\, ds, \tag{5}$$

where C^* is the transpose of the matrix C.

The first fact to be noted is that R_t is nonnegative definite and not necessarily positive definite, for if $v \in R^n$ is orthogonal to the subspace

$$[A, B] := \operatorname{span}_{R^n}\{A^{k-1}B\varepsilon_i \mid k = 1, \cdots, n,\ i = 1, \cdots, p,\ \varepsilon_i = (\delta_{i1}, \cdots, \delta_{ip})^*\},$$

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