The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 735–744

APPROACHABILITY IN A TWO-PERSON GAME

BY TIEN-FANG HOU

Bell Telephone Laboratories, Incorporated

1. Introduction. Let $M = ||M(i, j)||$ be an $r \times s$ matrix whose elements $M(i, j)$ are probability distributions with finite $E||\cdot||^\alpha$, $||\cdot||$ is the Euclidean norm and $\alpha > 1$, in a Euclidean k-space $\mathscr{E}^k$. We associate with M a game between two players, I and II, with the following infinite sequence of engagements: At the nth engagement, $n = 1, 2, \cdots$, player I selects $i = 1, \cdots, r$ with probability $p_n(1), \cdots, p_n(r)$, $\sum_{i=1}^r p_n(i) = 1$, and player II selects $j = 1, \cdots, s$ with probability $q_n(1), \cdots, q_n(s)$, $\sum_{j=1}^s q_n(j) = 1$. Each selection is made without either player knowing the choice of the other player. Having chosen i and j, payoff $Y_n \in \mathscr{E}^k$ is then determined according to the distribution $M(i, j)$. The point Y_n and probabilities

$$p_n = (p_n(1), \cdots, p_n(r)) \quad \text{and} \quad q_n = (q_n(1), \cdots, q_n(s)) \tag{1.1}$$

are announced to both players after each engagement. We call p_n player I's move and q_n player II's move.

A strategy for player I is a sequence of functions $f = \{f_n\}$, $n = 0, 1, 2, \cdots$, where f_n is defined on the $3n$-tuples $(p_1, q_1, Y_1; \cdots; p_n, q_n, Y_n)$ with value p_{n+1} in

$$P = \{p = (p(1), \cdots, p(r)) \colon \textstyle\sum_1^r p(i) = 1 \quad \text{and} \quad p(i) \geqq 0\}, \tag{1.2}$$

and $p_1 = f_0$ is simply a point of P. For player II, a strategy $g = \{g_n\}$ is defined similarly, except that

$$g_n(p_1, q_1, Y_1; \cdots; p_n, q_n, Y_n) = q_{n+1} \in Q \quad \text{and} \quad q_1 = g_0 \in Q, \tag{1.3}$$

where

$$Q = \{q = (q(1), \cdots, q(s)) \colon \textstyle\sum_1^s q(j) = 1 \quad \text{and} \quad q(j) \geqq 0\}. \tag{1.4}$$

For a given M, each strategy pair f, g determines a sequence of random variables $Y_1, Y_2, \cdots$ (vector payoffs) in $\mathscr{E}^k$.

Our objective here is to investigate the controllability of the center of gravity of the actual payoffs $\bar{Y}_n = \sum_1^n Y_m/n$ in a long series of plays.

We denote the Euclidean distance between $\bar{Y}_n$ and a nonempty set S in k-space by $\delta(\bar{Y}_n, S)$. For a given M, the set S is said to be approachable (see [1] and [2]) by I in M, if there exists an f^* for I such that, for every g,

$$\delta(\bar{Y}_n, S) \to 0 \quad \text{a.s.}, \tag{1.5}$$

where $Y_1, Y_2, \cdots$ are the payoffs determined by f^*, g. The set S is excludable by

Received August 15, 1969; revised October 26, 1970.