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THE DISTRIBUTION OF LINEAR COMBINATIONS OF ORDER STATISTICS FROM THE UNIFORM DISTRIBUTION

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1. Introduction. In this paper we derive an algorithm for computing the distribution function of an arbitrary linear combination of order statistics from a uniform distribution. Suppose $U_{(i)}$ is the ith smallest observation from a sample of size n from the uniform distribution on [0, 1], with the convention $U_0 \equiv 0$, $U_{n+1} \equiv 1$. Consider a set of S integers $\{k_i\}$ such that

$$k_0 = 0 < k_1 < k_2 < \cdots < k_S \leqq n. \tag{1.1}$$

For any set of constants $d_i > 0$ and any x, we seek

$$P\{\sum_{s=1}^{S} d_s U_{(k_s)} \leqq x\}.$$

Our approach is to generalize a formula derived by Dempster and Kleyle (1968).

2. Derivation of the algorithm. Let $X_i = U_{(i)} - U_{(i-1)}$, $i = 1, 2, \cdots n$. Let $c_{n+1} = 0$.

Define $c_1, c_2, \cdots c_n$ by

$$c_{k_i} = c_{k_i+1} + d_i \qquad \text{for} \quad i = 1, \cdots S$$

$$c_j = c_{j+1} \qquad \text{for} \quad j \notin (k_1, \cdots k_S).$$

Then we have

$$\sum_{s=1}^{S} d_s U_{(k_s)} = \sum_{i=1}^{n} c_i X_i. \tag{2.1}$$

For the special case $S = n$, Dempster and Kleyle (1968) have shown that

$$P\left\{\sum_{i=1}^{n} c_i X_i \leqq x\right\} = 1 - \sum_{j=1}^{r} \frac{(c_j - x)^n}{c_j \prod_{i \neq j} (c_j - c_i)} \tag{2.2}$$

for $0 \leqq x \leqq c_1$, where r is the largest positive integer such that $x \leqq c_r$. In the general case $S \leqq n$, we wish to allow

$$c_{k_{s-1}+1} = c_{k_{s-1}+2} = \cdots = c_{k_s} = c_{(s)},$$

for $s = 1, 2, \cdots S$.

Let $k_s - k_{s-1} = r_s$, $s = 1, \cdots S$; and $n - k_S = r_{S+1}$. Then we wish to let the first r_1 c_1's take the value $c_{(1)}$, the next r_2 take the value $c_{(2)}$, etc. Let $c_{(s+1)} = 0$. In this situation (2.2) is not applicable unless $r_s = 1$ for all s.

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