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PROJECTION WITH THE WRONG INNER PRODUCT AND ITS APPLICATION TO REGRESSION WITH CORRELATED ERRORS AND LINEAR FILTERING OF TIME SERIES

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1. Introduction. In many places in statistics one wants to calculate the orthogonal projection Px of some vector x on a subspace $\mathscr{P}$. Oftentimes the inner product function is specified by the unknown covariances C of a set of random variables. The usual procedure is to estimate C by C^* and approximate Px by P^*x, the orthogonal projection with respect to C^*; that is, x is projected on $\mathscr{P}$ using a wrong inner product. There is, therefore, interest in knowing when P^* will be a good approximation of P.

In Section 2, the question of calculating orthogonal projections with the wrong inner product in a general Hilbert space is investigated. The results are then applied to the problem of regression with correlated errors in Section 3 and to linear filtering operations on multi-channel, wide-sense stationary, stochastic processes in Section 4.

2. Projection with the wrong inner product in a general Hilbert space. Let $\mathscr{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\| = (\cdot, \cdot)^{\frac{1}{2}}$. Let $[\cdot, \cdot]$, which will be thought of as the wrong inner product, be a bilinear functional with the following properties:

(1) $[\cdot, \cdot]$ is defined on $\mathscr{D} \times \mathscr{D}$ where $\mathscr{D}$ is a linear subset of $\mathscr{H}$ whose closure is $\mathscr{H}$, $[x, y] = \overline{[y, x]}$, and for fixed x the linear functional $[x, \cdot]$ on $\mathscr{D}$ is bounded.

(2) If z_n is a sequence of vectors in $\mathscr{H}$ such that $z_n \to z$ and z_n is a $[\cdot, \cdot]$ Cauchy sequence, that is $[z_n - z_m, z_n - z_m] \to 0$ as $n, m \to \infty$, then $z \in \mathscr{D}$.

In (1), it has not been assumed that $[\cdot, \cdot]$ is a bounded bilinear functional. Assumption (2) has been made to ensure that $[\cdot, \cdot]$ is defined everywhere it is possible to do so and maintain the properties in (1).

From (1), for fixed $x \in \mathscr{D}$, the definition of the linear functional $[x, \cdot]$ may be extended boundedly to all of $\mathscr{H}$. From the Riesz Representation Theorem (Halmos, 1957, page 31) there exists $y \in \mathscr{H}$ such that $[x, \cdot] = (y, \cdot)$. Let B be the mapping defined by $Bx = y$. B is a linear and self-adjoint, and B is bounded if and only if $[\cdot, \cdot]$ is a bounded bilinear functional.

Let $\mathscr{P}$ be a subspace of $\mathscr{H}$ and P the orthogonal projection operator onto $\mathscr{P}$. Let $\mathscr{P}^* = \mathscr{P} \cap \mathscr{D}$ and let $\mathscr{Q}^*$ be the set of all $x \in \mathscr{D}$ such that $[x, y] = 0$ for all

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