The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 572–577

ONE-WAY EXPECTED UTILITY WITH FINITE CONSEQUENCE SPACES

BY PETER C. FISHBURN

Research Analysis Corporation

1. Introduction. Throughout this paper I shall assume that X is a nonempty finite set and that $\mathscr{P}$ is the set of probability distributions on X. The purpose of the paper is to examine critically conditions for a binary relation $\succ$ (preference) on $\mathscr{P}$ that imply

PROPOSITION 1. *There is a real-valued function u on X such that, for all* $P, Q \in \mathscr{P}$,

$$P \succ Q \Rightarrow E(u, P) > E(u, Q). \tag{1}$$

In (1), $E(u, P) = \sum u(x)P(x)$, the expected value of u under P.

The expected-utility representation in (1) can be thought of as unidimensional one-way ($\Rightarrow$) expected utility, in contrast to the unidimensional two-way ($\Leftrightarrow$) representation

$$P \succ Q \Leftrightarrow E(u, P) > E(u, Q) \tag{2}$$

that is implied by the von Neumann–Morgenstern axioms [8]. Our interest in (1) stems primarily from the fact that (1), unlike (2), does not imply that the relation (not $P \succ Q$, not $Q \succ P$) is transitive. For further comments on this point see Aumann [2] and Fishburn [4].

The one-way representation (1) has been studied previously by Aumann [2], [3] and Kannai [5]. Both remark on the difficulties encountered when X is allowed to be infinite, and Kannai investigates multidimensional expected-utility in this case. Although Aumann's formulation differs slightly from mine, there is no difficulty in translating his formulation into the one used here. In particular, the following theorem is similar to Aumann's Theorem A [2], and its proofs in ([4] Chapter 9) and in this paper are similar to Aumann's proof. [$\alpha P+(1-\alpha)R$ is the convex linear combination of P and R so that, for all $A \subseteq X$, $(\alpha P+(1-\alpha)R)(A) = \alpha P(A)+(1-\alpha)R(A)$. Recall that X is assumed to be finite.]

THEOREM 1. *Proposition* 1 *is true if the following four conditions hold throughout* $\mathscr{P}$:

A1. $\succ$ *is transitive.*

A2. $\alpha \in (0, 1)$ *and* $P \succ Q \Rightarrow \alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)R$.

A3. $\alpha \in (0, 1)$ *and* $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)R \Rightarrow P \succ Q$.

A4. $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$ *for all* $\alpha \in (0, 1] \Rightarrow$ *not* $S \succ R$.

To quote Aumann ([2] page 451), A2 "asserts that a preference is not changed by 'dilution'," and A3 says "that if we have a diluted preference, then the correspond-

Received December 1, 1969; revised September 10, 1970.