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WEAK CONVERGENCE RESULTS FOR A CLASS OF MULTIVARIATE MARKOV PROCESSES

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1. Introduction. In this paper we obtain weak convergence results for several variations and generalizations of the classical Ehrenfest process. The basic model considered here can be described as follows: N balls are distributed among K urns, with $n_k(t)$ balls being in urn k at time t. Balls move among the urns according to the following rules: The probability that a ball shifts from urn k to urn $l \neq k$ during $(t, t+\Delta t)$ is $n_k \lambda_k p_{kl} \Delta t + o(\Delta t)$, $(1 \leqq k, l \leqq K)$, and the probability of more than one transition during $(t, t+\Delta t)$ is of order $o(\Delta t)$. We are interested in the limiting behavior of suitably normalized versions of the process $(n_1(t), \cdots, n_K(t))$, as the number of balls gets large. Our main result states that, suitably normalized, these processes converge to what we call multivariate Ornstein–Uhlenbeck processes in the sense of weak convergence of probability measures [2]. In particular, we arrive at a diffusion approximation for $(n_1(t), \cdots, n_K(t))$ as N gets large.

The technique of a "random change of time" enables us to obtain weak convergence results for a large class of discrete-time multivariate Ehrenfest models also. As a special case, we obtain a new proof of Iglehardt's [5] limit theorems.

A diffusion approximation to the two-urn model has been derived by Kac [6]. Karlin and McGregor [7] analyze several multivariate extensions of those results. Iglehardt's limit theorems are closest to ours, but they require that the time-parameter be discrete and that the probabilities p_{kl} depend on l only.

Several applications of multivariate Ehrenfest urn-type models are mentioned in [5], page 875. The model has also been found useful in describing the distribution of N vehicles over the K lanes of a long stretch of a unidirectional K-lane freeway [9]. In this application variables of interest are the usage of different lanes, the return to equilibrium after a bottleneck situation, etc.

Weak convergence results can be useful in two directions: For large N the distribution of a certain functional of the process can be approximated by the distribution of the same functional of the limiting process, provided that the latter is known. On the other hand, if an approximate distribution of a functional under the limiting process is desired, it might be possible to obtain the distribution of the same functional under the Nth approximating process, where N is large. In our case this procedure is particularly useful if simulation techniques are to be employed. Since the approximating processes have sample paths which are step

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