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A NOTE ON THE ARC-SINE LAW AND MARKOV RANDOM SETS

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0. Introduction. The purpose of this note is to point out an extension of a theorem of Dynkin [2], [3, page 447] concerning renewal processes to the context of Markov random sets (or, as we shall call them, *semilinear Markov processes*). Dynkin's result states that, if $X_1, X_2, \cdots$ is a "renewal sequence", i.e. a sequence of nonnegative, independent, identically distributed random variables, with partial sums S_n, and if x_t is defined by

$$x_t = t - \max\{S_n : S_n \leqq t\}, \qquad t \geqq 0, \tag{0.1}$$

then x_t/t has a nondegenerate limiting distribution as $t \to \infty$ iff $1-F(x) = x^{-\beta}L(x)$, where $F(x)$ is the common distribution function of the X_i, β is some number in $(0, 1)$, and $L(x)$ is slowly varying as $x \to \infty$.

Semilinear Markov processes arise when, in (0.1), we allow more general processes with stationary, independent increments. Specifically, let $T(s)$, $s \geqq 0$, be a subordinator (terminology is explained in Section 1) on a probability space $(\Omega, \mathscr{F}, P)$ having exponent $g(\lambda) = \lambda\alpha + \int_0^\infty (1-e^{-\lambda y})\mu(dy)$. Denote by $Q(\omega)$ the range of $T(s, \omega)$, $s \geqq 0$, and define the random function $\xi_t(\omega)$ by

$$\xi_t(\omega) = t - \sup\{u \leqq t : u \in Q(\omega)\}, \qquad t \geqq 0. \tag{0.2}$$

This is analogous to (0.1). Our extension of Dynkin's theorem may now be stated as follows.

0.3. THEOREM. *Let* $h(x) = \mu(x, \infty]$. *If* $h(x) = x^{-\beta}L(x)$, *where* $0 < \beta < 1$ *and* $L(x)$ *is of slow variation as* $x \to \infty$, *then* ξ_t/t *has a limiting distribution as* $t \to \infty$, *given by the measure*

$$\nu(dx) = \frac{\sin \pi\beta}{\pi} x^{-\beta}(1-x)^{\beta-1}\,dx \tag{0.4}$$

on $(0, 1)$.

The converse is a bit more delicate and is treated in Section 3. In particular, the possible degenerate limit laws are completely delineated, a result which appears to be new even in the case studied by Dynkin. The measure ν is the "generalized arc-sine" distribution [3, page 446] which arises, among other places, in the theory of semi-stable Markov processes [8, page 68], [4, Section 4]. There are also some closely related results in [7], and in [9] see, especially, Theorem 8.1 (it is not known if our result follows therefrom; in any case the present method of proof should be of independent interest).

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