The Annals of Mathematical Statistics
1971, Vol. 42, No. 4, 1379–1402

FORMAL BAYES ESTIMATION WITH APPLICATION TO A RANDOM EFFECTS MODEL

BY STEPHEN PORTNOY
Harvard University

1. Introduction. In this paper we will consider some techniques of formal Bayes estimation and apply them to the estimation of components of variance in the one way layout random effects model of the analysis of variance. In particular, we will consider the following problem in canonical form: we observe

$$\bar{Y} \sim \mathfrak{N}(\mu, (\sigma_e + J\sigma_a)/IJ), S_1 \sim \sigma_e \chi^2_{I(J-1)}, \quad \text{and} \quad S_2 \sim (\sigma_e + J\sigma_a)\chi^2_{I-1}$$

where I and J are positive integers (the number of treatments and replications respectively), μ is real, and σ_e and σ_a are positive. We want to find estimates for σ_e and σ_a using essentially mean squared error as a measure of performance.

The problem of estimating σ_a and σ_e is not new, and minimum variance unbiased estimates and maximum likelihood estimates are well known. However, one can look for improvements; and in the estimation of σ_a a special problem arises; the unbiased estimate may be negative and the maximum likelihood estimate may be exactly zero. This particular problem has been considered recently by a number of investigators (see, for example, Scheffé [12] and Thompson [18]) and various interpretations for such estimates have been suggested. However, in problems where estimates are really desired, use of such estimators seems to me to be unacceptable. To solve this problem we will consider the use of formal Bayes estimators (i.e. Bayes estimators versus priors which are not necessarily finite), which will be strictly positive. We will show that certain formal Bayes estimators both of σ_a and σ_e have good mean squared error properties and can seriously be recommended.

Inferences about σ_e and σ_a from a Bayesian viewpoint have been recently presented by Box and Tiao [1], Hill [4], Stone and Springer [17], and Tiao and Tan [19]. The methods described in these papers are generally based on use of the Jeffreys' prior. We will later compare these methods with ones considered here and give reasons why the present methods should be generally preferable.

The techniques used here are special cases of more general considerations applicable whenever the statistical problem is invariant under a group of transformations which does not act transitively on the parameter space (i.e. in problems where there is not a unique best invariant procedure). The analysis of variance problem considered here is easily seen to be invariant under location and scale changes (if invariant loss functions are used); that is, under transformations on the parameter space $\{(\mu, \sigma_e, \sigma_a): -\infty < \mu < \infty, \sigma_e > 0, \sigma_a > 0\}$ described by

$$\mu \to a\mu + b, \sigma_e \to a^2\sigma_e, \sigma_a \to a^2\sigma_a.$$

Received January 19, 1970; revised February 8, 1971.