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A NOTE ON THE WEAK CONVERGENCE OF STOCHASTIC PROCESSES[1]

BY M. J. WICHURA
University of Chicago

A simple method giving quick access to some important general weak convergence theorems is described and illustrated.

1. Introduction. We point out in this note a quick and easy method for establishing some general weak convergence theorems for probability distributions on a metric space. The only prerequisite needed is the so-called "portmanteau" theorem of Billingsley (1968), which gives various equivalent descriptions of weak convergence. More powerful methods are available and frequently yield more information. For instance, the method of Prohorov (1956), based on a characterization of the relative compactness of a family of probability distributions, enables one to deduce the existence of (as well as the weak convergence to) the limit distribution in the applications given below. The method of Skorohod (1956), which, roughly speaking, replaces a given sequence of processes having weakly convergent finite-dimensional distributions by a new sequence of processes having the same laws as the old processes but with sample paths converging almost surely at each time point, enables one to apply standard analytic arguments concerning pointwise convergent functions. But both these methods require a fair bit of effort for their development and application. A simple procedure that allows one to get into the thick of things quickly is therefore of some interest. Such a procedure is described and illustrated in what follows.

2. The method. Let (S, d) be a metric space, and let $\mathscr{S}$ be its Borel σ-algebra. Let X and X_k $(k \geqq 1)$ be S-valued random variables (measurable with respect to $\mathscr{S}$), each defined on some probability space. One says that the X_k's converge in distribution to X, and writes $X = d\lim_k X_k$, if $Ef(X_k) \to Ef(X)$ for each continuous bounded real-valued function f on S. The following result states that $X = d\lim_k X_k$ provided one has convergence in distribution for sufficiently small perturbations of the original variables.

PROPOSITION 1. *Let S, d, X, and X_k be as described above. For each $n \geqq 1$, let $A_n : S \to S$ be $\mathscr{S}$-measurable. Suppose that*

$$d\lim_k A_n X_k = A_n X \qquad \textit{for each } n \tag{1}$$

$$\text{plim}_n \lim_k d(X_k, A_n X_k) = 0 \tag{2}$$

$$\text{plim}_n d(X, A_n X) = 0. \tag{3}$$

Then $X = d\lim_k X_k$.

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