The Annals of Mathematical Statistics
1971, Vol. 42, No. 6, 2156–2158

COMPLETION OF A DOMINATED ERGODIC THEOREM

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In 1937 [1], Marcinkiewicz and Zygmund proved that for $r \geqq 1$, independent, identically distributed (i.i.d.) random variables $\{X_n, n \geqq 1\}$ satisfy

$$E \sup_{n \geqq 1} n^{-r} |\textstyle\sum_1^n X_i|^r < \infty \tag{1}$$

provided

$$E|X_1|^r < \infty, r > 1 \quad \text{and} \quad E|X_1|^r \log^+ |X_1| < \infty, r = 1. \tag{2}$$

In the following year Wiener [4] demonstrated the analogous result in the more general context of measure—preserving transformations—and this as well as subsequent operator generalizations have come to be known as dominated ergodic theorems.

Reverting to the i.i.d. case, it was proved in 1967 [3] that if, in addition the rv's satisfy $EX_1 = 0$, then for $r \geqq 2$ (this restriction is necessary)

$$E \sup_{n \geqq 1} c_n |\textstyle\sum_1^n X_i|^r < \infty \tag{3}$$

for c_n such as $n^{-r/2}(\log n)^{-(r/2k)-\delta}$ with $\delta > 0$ and $k =$ greatest integer $\leqq r$ provided $r = 2$ plays the role of $r = 1$ in condition (2). A major step forward was taken by Siegmund [2] who proved the theorem below for *integral values*[2] *of* r. It is the purpose of this note to complete the analogy with the result of Marcinkiewicz and Zygmund by proving the theorem for non-integral values of r as well. This is accomplished by modification of an idea of [3] in conjunction with the approach of [2].

Theorem. *For* $r \geqq 2$, *independent, identically distributed random variables* $\{X_n, n \geqq 1\}$ *with* $EX_1 = 0$ *satisfy*

$$E \sup_{n \geqq e^e} \frac{|\sum_1^n X_i|^r}{(n \log \log n)^{r/2}} < \infty \tag{4}$$

if and only if

$$E|X_1|^r < \infty, r > 2 \quad \text{and} \quad E\,\frac{X_1^2 \log |X_1|}{\log \log |X_1|} I_{[|X_1| > e^e]} < \infty, \qquad r = 2.$$

The proof of the theorem will be facilitated by the following proposition which may have independent interest.

Received October 21, 1970.
[1] Research under NSF contract GP–9301.

[2] Strictly speaking, the theorem is proved explicitly for $r = 2$ and stated for $r = 3, 4, \cdots$.