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GENERALIZATIONS OF THE GLIVENKO-CANTELLI THEOREM

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0. Introduction. Let μ be a probability measure on the Borel sets, $\mathscr{B}$, in k dimensional Euclidean space E_k and $X_1, X_2, \cdots$ a sequence of independent random vectors with values in E_k such that $P[X_i \in A] = \mu(A)$ for every A in $\mathscr{B}$, $i = 1, 2, \cdots$. A necessary and sufficient condition is given for

$$(*) \qquad \sup_{f \in \mathscr{M}} \left| n^{-1} \sum_{i=1}^{n} f(X_i) - \int f \, d\mu \right| \to_{a.s.} 0,$$

where $\mathscr{M}$ is the class of all monotone functions on E_k with a uniform bound. (*) is shown to hold with no restriction on μ for several classes of functions, one of which is the class of characteristic functions of half-spaces in E_k. This result strengthens the theorem of Wolfowitz (1954). I am obliged to H. D. Brunk for many invaluable conversations.

1. Sufficient conditions on $\mathscr{M}$ and μ for (*). Let $\mathscr{M}$ denote a class of real-valued, measurable, uniformly bounded functions defined on E_k. For f in $\mathscr{M}$ let

$$S_n(f) = n^{-1} \sum_{i=1}^{n} f(X_i).$$

If $\mathscr{M}$ and μ are such that

$$P[\lim_{n\to\infty} \sup_{f \in \mathscr{M}} |S_n((f) - \int f \, d\mu| = 0] = 1,$$

it will be said that (*) holds. It will be assumed that $\mathscr{M}$ is such that $\sup_{f \in} \; S_n(f)$ is measurable. The particular classes $\mathscr{M}$ discussed in Section 2 have this property.

Lemmas 1–6 give sufficient conditions on $\mathscr{M}$ and μ for (*), while the remainder of the results are concerned with (*) holding for particular classes $\mathscr{M}$.

LEMMA 1. *If corresponding to each positive number ε there is a finite class of functions $\mathscr{M}_\varepsilon$ such that for each f in $\mathscr{M}$ there are f_1 and f_2 in $\mathscr{M}_\varepsilon$ with $f_1 \leqq f \leqq f_2$ and $\int f_2 d\mu - \int_1 f d\mu < \varepsilon$, then (*) holds.*

PROOF. Corresponding to each positive integer k, let $\{f_1^{\,k}, f_2^{\,k}, \cdots, f^k{}_m\}$ be the finite class $\mathscr{M}_{1/k}$ which corresponds to the positive number $1/k$ by the hypothesis. If

$$A_i^k = [S_n(f_i^k) \to \int f_i^k \, d\mu] \qquad i = 1, 2, \cdots, m; k = 1, 2, \cdots,$$

then

$$P(A_i^k) = 1,$$

by the Law of Large Numbers.
If $A = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} A_i^k$, then $PA = 1$.

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