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A NOTE ON HARMONIC FUNCTIONS AND MARTINGALES[1]

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1. A decomposition theorem. In this note we will be concerned with the problem of decomposing a positive harmonic function h into a sum of three positive harmonic functions h_1, h_2, and h_3, each of which behaves quite differently when composed with Brownian motion. This problem has been treated in a very general context by Blumenthal and Getoor (1968) and, in fact, our main result (Theorem 1) is contained in Theorem 5.14 of Chapter IV. We present here a direct treatment based on the theory of conditional Brownian motion which, in addition to giving the required decomposition, characterizes the functions h_1, h_2 and h_3 in terms of their Martin boundary representations (Corollary 1). As will be seen in the examples, this last characterization is useful in understanding the nature of the non-uniformly integrable martingale component h_2. It is assumed throughout that the reader is familiar with the relationship between harmonic functions and Brownian motion as described in Doob (1954) and (1957a), and with the theory of the Martin boundary and conditional processes as developed in Doob (1957b) and (1958). Before stating the main result, we introduce some notation and recall a few facts.

Let D be a domain in n-dimensional Euclidean space which has a Green's function g. Let ∂D denote the Martin boundary of D and let ∂D_e denote the subset of ∂D consisting of the minimal points. $K(\eta, \cdot)$ will denote the minimal harmonic function associated with $\eta \in \partial D_e$ which is normalized so that $K(\eta, \xi_0) = 1$ for a fixed $\xi_0 \in D$.

Let Ω be the function space consisting of all continuous functions $\omega:[0, \infty) \to D \cup \partial D_e$ with the property that, if $\omega(s) = \eta \in \partial D_e$, then $\omega(t) = \eta$ for all $t \geqq s$. $X(t)$ will denote the tth coordinate function on Ω. Using the notation of Blumenthal and Getoor (1968), let $(\Omega, \mathscr{F}, \mathscr{F}(t), X(t), \theta(t), P_\xi)$ denote the standard Brownian motion process on D, stopped when ∂D_e is hit. Note that we can write P_ξ (or E_ξ) for $\xi \in D \cup \partial D_e$, but that each point of ∂D_e acts as an absorbing point. Define the lifetime τ by the equation

$$\tau(\omega) = \inf\{t: X(t) \in \partial D_e\}.$$

If h is a positive harmonic function on D, then $\lim_{t\uparrow\tau} h[X(t)]$ exists P_ξ-almost everywhere ($\xi \in D$) and, in fact, h defines a Borel measurable boundary function (which we continue to denote by h) on ∂D_e such that

$$\lim_{t\uparrow\tau} h[X(t)] = h[\lim_{t\uparrow\tau} X(t)] \tag{1}$$

P_ξ-almost everywhere ($\xi \in D$). When dealing with a given measure $P_\xi(\xi \in D)$, we adopt the convention that $h[X(t)]$ equals the quantity in (1) if $t \geqq \tau$. With this

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