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BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS, II: LIMIT THEOREMS[1]

BY KRISHNA B. ATHREYA AND SAMUEL KARLIN

University of Wisconsin and Stanford University

0. Introduction. We refer to [1] for the basic set up, notation and terminology. In this sequel to [1] we elaborate several analogs of the known limit theorems for simple Galton-Watson processes.

The B.P.R.E. will be labeled supercritical, critical, or subcritical according as $E \log \varphi'_{\zeta_0}(1) > 0$, $= 0$ or < 0 respectively. The supercritical case is distinguished in that extinction of the population is not a certain event for almost every realization of the environmental process. Here the classical Martingale theorem has a natural extension (see Theorem 1 of Section 1). In order to justify the rest of the terminology separating the critical from the subcritical case, we recall the following facts concerning one type Galton-Watson processes.

A subcritical simple branching process has the property that

$$\lim_{n\to\infty} P\{Z_n = k \mid Z_n \neq 0\} = a_k, \qquad k = 1, 2, \cdots \tag{1}$$

exists and $\{a_k\}$ determines a genuine discrete probability density while in the critical case the limit in (1) identically vanishes. In fact, in the latter case $E(Z_n \mid Z_n \neq 0) \sim cn$ provided $\varphi''(1) < \infty$ where $\varphi(s)$ is the progeny p.g.f. of the process. (Here c is an appropriate positive constant.) More specifically, we have the limit law

$$E\left(\exp\left[-\lambda \frac{Z_n}{n}\right] \middle| Z_n \neq 0\right) \to \frac{1}{1+\lambda a} \tag{2}$$

for suitable $a > 0$. This is commonly known as Kolmogorov's limit law while priority for (1) is generally attributed to Yaglom.

In order to develop a version of (1) in the context of B.P.R.E. we impose additional conditions on the environmental process $\{\zeta_t, t \geqq 0\}$.

DEFINITION 1. The stationary ergodic process ζ_t is said to be *exchangeable* if the vector random variables $(\zeta_i, \zeta_{i+1}, \cdots, \zeta_{i+n})$ and $(\zeta_{n+i}, \zeta_{n+i-1}, \cdots, \zeta_i)$ are identically distributed for each $i \geqq 0$ and $n \geqq 0$.

When ζ_t, $t \geqq 0$ consists of i.i.d. random variables then ζ_t is manifestly an exchangeable process. Another example of an exchangeable process arises when $(\zeta_t, t \geqq 0)$ is a stationary reversible ergodic Markov chain. Our result here is that the sequence of random probability distributions viz. $\{Z_n \mid Z_n \neq 0, \zeta\}$ converges in law to a random probability distribution. More precisely,

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