The Annals of Mathematical Statistics
1972, Vol. 43, No. 1, 205-229

CHARACTERIZATIONS OF THE MULTIVARIATE NORMAL DISTRIBUTION USING REGRESSION PROPERTIES[1]

BY F. S. GORDON[2] AND A. M. MATHAI

C. W. Post College and McGill University

1. Introduction. In a previous paper [2], the authors obtained a series of characterizations for many univariate distributions using the property of cubic regression for a cubic statistic on a linear one. These characterizations were obtained by solving a differential equation in the characteristic function of an arbitrary population.

In order to obtain characterizations of multivariate distributions analogous to those determined in the univariate case, it is necessary to utilize a concept for the derivative of any quantity, be it scalar, vector or matrix, with respect to the vector variable $t = [t_1, \cdots, t_p]$. This derivative is similar to that presented, for example, by Wedderburn [7]. On the basis of several properties which this derivative possesses, a vector differential equation is obtained in the characteristic function using the assumption of cubic regression for a cubic statistic on a linear one. For appropriate conditions on the coefficients of this equation, a series of characterizations for the multivariate normal distribution is obtained within the class of those populations whose characteristic functions can be expressed in a certain type of infinite series expansion. This expansion is one in the vector variable t and its transpose t', with coefficients which are determined in terms of the vector derivatives of the characteristic function. In particular, in the univariate case ($p = 1$), it is shown that this infinite series reduces to the usual Taylor series expansion about the origin and therefore, the corresponding univariate class of populations includes all those whose characteristic functions are analytic.

Throughout the present work, the symbol 0 is used to represent the scalar zero element, the zero vector and the null matrix. The particular usage is always clear from the context.

2. Generalized differentiation with respect to a vector variable. We begin by defining the derivative of any scalar, any $p \times 1$ vector, or any $p \times q$ matrix with respect to the vector variable

$$t = [t_1, t_2, \cdots, t_p] .$$

Received June 24, 1968.

[1] This paper is adapted from a portion of the first author's doctoral dissertation written under the supervision of Professor A. M. Mathai and submitted to the Faculty of Graduate Studies and Research, McGill University, 1968.

[2] The author wishes to acknowledge support of the National Research Council of Canada.