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A NOTE ON INFINITELY DIVISIBLE RANDOM VECTORS

BY MOSHE POLLAK
The Hebrew University

The purpose of this note is to show a characterization of infinitely divisible random vectors whose projection onto a subspace is normally distributed.

In this paper, $\mathbf{x}$ will denote a k-dimensional vector of real numbers, $\mathbf{x}'$ its transpose, $|\mathbf{x}|$ its length, $\boldsymbol{\xi}'\mathbf{x}$ the inner product of the vectors $\boldsymbol{\xi}$, $\mathbf{x}$, R^k the k-dimensional Euclidean space. By a normal distribution we understand either a proper or a singular normal distribution.

A nonnegative measure M defined on R^k is said to be canonical if it has no atom at the origin and $\int_{R^k} [1 + |\mathbf{x}|^2]^{-1}\, dM(\mathbf{x}) < \infty$.

For $\mathbf{X}$ to be an infinitely divisible random vector, it is necessary and sufficient that there exist a canonical measure M, a vector of real numbers $\mathbf{b}$, and a nonnegative definite symmetric matrix $\sum = (\sigma_{ij})$ such that

$$\text{(1)} \qquad \log Ee^{i\boldsymbol{\xi}'\mathbf{X}} = \int_{R^k} \frac{e^{i\boldsymbol{\xi}'\mathbf{x}} - 1 - i\boldsymbol{\xi}'\boldsymbol{\tau}(\mathbf{x})}{|\mathbf{x}|^2}\, dM(\mathbf{x}) + i\boldsymbol{\xi}'\mathbf{b} - \tfrac{1}{2}\boldsymbol{\xi}' \sum \boldsymbol{\xi}$$

where

$$\begin{aligned} \tau(x) &= -1 & x &\leqq -1 \\ &= x & -1 &\leqq x \leqq +1\,, \qquad [\boldsymbol{\tau}(\mathbf{x})]' = (\tau(x_1), \cdots, \tau(x_k))\,. \\ &= 1 & 1 &\leqq x \end{aligned}$$

This representation is unique (see [1] page 559).

Notice that if $\mathbf{X}$ is normal, $\log Ee^{i\boldsymbol{\xi}'\mathbf{X}} = i\boldsymbol{\xi}'\mathbf{b} - \frac{1}{2}\boldsymbol{\xi}' \sum \boldsymbol{\xi}$, and thus for normally distributed random vectors M is identically zero.

THEOREM. *Let* $\mathbf{X}' = (X_1, \cdots, X_k)$ *be an infinitely divisible random vector, and* $\boldsymbol{\xi}_0 \neq \mathbf{0}$ *an arbitrary vector in k-space. The distribution of* $\boldsymbol{\xi}_0'\mathbf{X}$ *is one-dimensional normal if and only if the canonical measure* M *which corresponds to* $\mathbf{X}$ *in the representation* (1) *is carried by the subspace* $\{\mathbf{x} \mid \boldsymbol{\xi}_0'\mathbf{x} = 0\}$.

PROOF. We need only prove that if $\boldsymbol{\xi}_0'\mathbf{X}$ is normal, then M is carried by $\{\mathbf{x} \mid \boldsymbol{\xi}_0'\mathbf{x} = 0\}$.

Since for every infinitely divisible random vector $\mathbf{X}$ and every matrix A, $A\mathbf{X}$ is infinitely divisible, we may assume without loss of generality that $\boldsymbol{\xi}_0' = (1, 0, \cdots, 0)$. Then:

$$\text{(2)} \qquad \log Ee^{it\boldsymbol{\xi}_0'\mathbf{X}} = ib_1 t - \tfrac{1}{2}t^2\sigma_{11} + \int_{R^k} \frac{e^{itx_1} - 1 - it\tau(x_1)}{|\mathbf{x}|^2}\, dM(\mathbf{x})$$

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