

## NOTE ON A "KOLMOGOROV-TYPE" INEQUALITY

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An inequality, described as similar to Kolmogorov's but without the assumption of independence, was proved in [1] and accompanied by several corollaries (cf. also the highly favorable review #6370, *Math. Reviews* 39 (1970)). The purpose of this note is to give a much shorter proof of the inequality and to point out that the corollaries are all trivial even without using the inequality. The reader may also easily check that Loève's [2] page 241–242 propositions, implying the corollaries, are *not* trivial.

To make the note shorter we shall limit ourselves to the case of Theorem 1, [1] when  $0 < r \leq 1$  and thus  $(a + b)^r \leq a^r + b^r$  for all nonnegative numbers  $a, b$ . (For Theorem 2, when  $r > 1$ , one would use the Minkowski inequality.) Now let  $X_i$  be random variables,  $0 < r \leq 1$ ,  $\varepsilon > 0$  and  $\{c_i\}$  a non-increasing sequence of positive numbers. Then the result of Theorem 1 can be formally strengthened, by replacing  $X_i$  by  $|X_i|$ , to

$$(*) \quad P\{\max_{m \leq k \leq n} c_k \sum_1^k |X_i| \geq \varepsilon\} \leq \varepsilon^{-r} [c_m^r \sum_1^m E|X_i|^r + \sum_{m+1}^n c_i^r E|X_i|^r].$$

But it is easy to verify that the random variable appearing at the left-hand side is less or equal to  $c_m \sum_{i=1}^m |X_i| + \sum_{i=m+1}^n c_i |X_i|$  which has the  $r$ th moment less or equal  $\varepsilon^r$  times the right-hand side in (\*). The Markov inequality yields (\*).

As for the corollaries take Corollary 2 as a typical example: If  $E|X_i| = v < +\infty$  and  $q > 0$  then  $\sum_1^\infty |X_i|/i^{q+1}$  is integrable and thus is finite a.e. The Kronecker lemma gives then the desired result  $n^{-q-1} \sum_1^n X_i \rightarrow 0$  a.e. The example following Corollary 3 follows immediately from the Borel-Cantelli lemma.

### REFERENCES

- [1] KOUNIAS, EUSTRATIOS G. and WENG, TENG-SHAN (1969). An inequality and almost sure convergence. *Ann. Math. Statist.* 40 1091–1093.  
[2] LOÈVE, M. (1955). *Probability Theory*. Van Nostrand, Princeton.

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