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CHOOSING A POINT FROM THE SURFACE OF A SPHERE

BY GEORGE MARSAGLIA

McGill University

A frequent problem in Monte Carlo is that of sampling uniformly from the surface of the unit 3-sphere $\{(z_1, z_2, z_3): z_1^2 + z_2^2 + z_3^2 = 1\}$. Such problems arise in connection with random rotations, orientations, directions. Refer to *random directions* in the index of Feller **2** [2] for a number of applications, and to Stephens [4], or Watson and Williams [5], for references on statistical problems associated with uniform distributions on a sphere. It is an interesting problem in probability theory to find functions of uniform variates which produce a point with uniform distribution on the surface of the sphere, and which are practical for use in computers. There are three well-known methods, two of them obvious and the other, by J. M. Cook [1], based on elegant theory but unfortunately too slow to be practical. I will summarize the three methods here and then offer a new method that is simpler and about twice as fast as any of the existing methods.

Method 1. Generate V_1, V_2, V_3, independent uniform on $(-1, 1)$ until $S = V_1^2 + V_2^2 + V_3^2 < 1$ then form

$$(V_1/S^{\frac{1}{2}}, V_2/S^{\frac{1}{2}}, V_3/S^{\frac{1}{2}}) . \tag{1}$$

The idea here is to choose a point in a cube, reject it unless it is in the inscribed sphere, then project the point to the surface of the sphere. Efficiency is $\pi/6$, so the method requires an average of $18/\pi \cong 5.73$ uniform variates.

Method 2. Generate X_1, X_2, X_3, independent standard normal variates, put $S = X_1^2 + X_2^2 + X_3^2$ and form

$$(X_1/S^{\frac{1}{2}}, X_2/S^{\frac{1}{2}}, X_3/S^{\frac{1}{2}}) . \tag{2}$$

This method is obvious to probabilists, but perhaps because of the importance of the problem of sampling from the surface of a sphere, was the subject of a paper by Muller [3]. (Not so obvious is the converse: if independent X_1, X_2, X_3 lead to (2) with a uniform distribution on the sphere, what can be said about the distributions of the X's?) Method 2 is slower than Method 1, and even in higher dimensions there are better methods for getting a point on the n-sphere. (Method 1 is hopeless in higher dimensions.)

Method 3 (Cook, [1]). Generate V_1, V_2, V_3, V_4 independent uniform on $(-1, 1)$ until $S = V_1^2 + V_2^2 + V_3^2 + V_4^2 < 1$, then form

$$(2(V_2V_4 + V_1V_3)/S, 2(V_3V_4 - V_1V_2)/S, (V_1^2 + V_4^2 - V_2^2 - V_3^2)/S) . \tag{3}$$

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