CHOOSING A POINT FROM THE SURFACE OF A SPHERE

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A frequent problem in Monte Carlo is that of sampling uniformly from the surface of the unit 3-sphere \( \{(z_1, z_2, z_3): z_1^2 + z_2^2 + z_3^2 = 1\} \). Such problems arise in connection with random rotations, orientations, directions. Refer to random directions in the index of Feller 2 [2] for a number of applications, and to Stephens [4], or Watson and Williams [5], for references on statistical problems associated with uniform distributions on a sphere. It is an interesting problem in probability theory to find functions of uniform variates which produce a point with uniform distribution on the surface of the sphere, and which are practical for use in computers. There are three well-known methods, two of them obvious and the other, by J. M. Cook [1], based on elegant theory but unfortunately too slow to be practical. I will summarize the three methods here and then offer a new method that is simpler and about twice as fast as any of the existing methods.

Method 1. Generate \( V_1, V_2, V_3 \), independent uniform on \((-1, 1)\) until \( S = V_1^2 + V_2^2 + V_3^2 < 1 \) then form

\[
(\frac{V_1}{S^1}, \frac{V_2}{S^1}, \frac{V_3}{S^1}).
\]

The idea here is to choose a point in a cube, reject it unless it is in the inscribed sphere, then project the point to the surface of the sphere. Efficiency is \( \pi/6 \), so the method requires an average of \( 18/\pi \approx 5.73 \) uniform variates.

Method 2. Generate \( X_1, X_2, X_3 \), independent standard normal variates, put \( S = X_1^2 + X_2^2 + X_3^2 \) and form

\[
(\frac{X_1}{S^1}, \frac{X_2}{S^1}, \frac{X_3}{S^1}).
\]

This method is obvious to probabilists, but perhaps because of the importance of the problem of sampling from the surface of a sphere, was the subject of a paper by Muller [3]. (Not so obvious is the converse: if independent \( X_1, X_2, X_3 \) lead to (2) with a uniform distribution on the sphere, what can be said about the distributions of the \( X \)'s?) Method 2 is slower than Method 1, and even in higher dimensions there are better methods for getting a point on the \( n \)-sphere. (Method 1 is hopeless in higher dimensions.)

Method 3 (Cook, [1]). Generate \( V_1, V_2, V_3, V_4 \) independent uniform on \((-1, 1)\) until \( S = V_1^2 + V_2^2 + V_3^2 + V_4^2 < 1 \), then form

\[
\frac{2(V_2v_4 + V_4v_3)/S, 2(V_3v_4 - V_1v_2)/S, (V_1^2 + V_4^2 - V_2^2 - V_3^2)/S}{(V_2^2 + V_3^2 + V_4^2)/S}.
\]

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