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# STATISTICAL STRUCTURE OF THE PROBLEM OF SAMPLING FROM FINITE POPULATIONS

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**1. Introduction.** Let $X$ be a space, $\mathbf{A}$ a $\sigma$-field of subsets of $X$, and $P = \{p\}$ a family of probability measures on $\mathbf{A}$. A triplet $(X, \mathbf{A}, P)$ is often called a statistical structure. Basu and Ghosh [1] proposed as a measure-theoretic expression of the statistical problems of sampling from finite populations a special type of statistical structure satisfying the following assumptions.

ASSUMPTIONS. (1) $X$ is a space containing more than countably many points.
(2) $\mathbf{A}$ is the $\sigma$-field consisting of all subsets of $X$.
(3) Each $p \in P$ is a discrete probability measure.
(4) $p(A) = 0$ for all $p \in P$ implies $A = \emptyset$.

Section 2 of the present paper gives a few results about this structure. It is proved that a $\sigma$-field is inducible if and only if it is also a complete field, that is, it is closed under the formation of arbitrarily many (possibly more than countable) number of sets in it. It is also shown that the pairwise sufficiency of an inducible $\sigma$-field (a statistic) implies its sufficiency. Furthermore, the essential completeness of the class of tests which are measurable with respect to an inducible $\sigma$-field implies its sufficiency. It is observed that these results do not generally hold for non-inducible $\sigma$-fields. A partial analogue of Neyman's factorization theorem which characterizes pairwise-sufficient $\sigma$-fields is given.

Section 3 contains an attempt, based on the results in Section 2, to remove some inconveniences that are still remaining in the same structure.

**2. Sufficiency and pairwise-sufficiency.** We use here essentially the same definitions and similar notations as in [1]. Thus by a *statistic* we mean a *partition* $\mathbf{T} = \{T\}$, a class of mutually disjoint non-empty subsets of $X$ which collectively cover $X$. Two partitions $\mathbf{T}$ and $\mathbf{U}$ are written $\mathbf{T} > \mathbf{U}$ when every set in $\mathbf{U}$ is a union of some sets in $\mathbf{T}$. A letter $\mathbf{B}$ will usually stand for a sub-$\sigma$-field of $\mathbf{A}$. Any statistic $\mathbf{T}$ induces a sub-$\sigma$-field $\mathbf{B}(\mathbf{T})$ of $\mathbf{A}$, the class of all those sets in $\mathbf{A}$ which can be expressed as a union of sets in $\mathbf{T}$. Two points $y$ and $z$ in $X$ belonging to a common $T$ in $\mathbf{T}$ are written $y \sim z(\mathbf{T})$. This is an equivalence relation on $X$, and conversely any equivalence relation gives rise to a statistic defined as the totality of its equivalence classes. On the other hand, $y \sim z(\mathbf{B})$ means that each set $B$ in $\mathbf{B}$ either contains both or neither of $y$ and $z$. This,

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