The Annals of Mathematical Statistics
1972, Vol. 43, No. 4, 1155-1163

ON BAHADUR EFFICIENCY OF THE JOINT-RANKING PROCEDURE[1]

BY K. L. MEHRA

The University of Alberta

1. Introduction. Consider the problem of testing the equality of several, say K, treatments on the basis of paired-observations, viz. (X_{il}, X_{jl}), $l = 1, 2, \cdots, N_{ij}$ $(1 \leqq i < j \leqq K)$ obtained by N_{ij} independent paired-comparisons for each pair (i, j) of treatments. If we assume that the N_{ij} differences $Z_l^{(i,j)} = X_{il} - X_{jl}$, $l = 1, 2, \cdots, N_{ij}$ have a common continuous cdf $G_{ij}(z)$ $(1 \leqq i < j \leqq K)$, the hypothesis of no-difference among the treatments can be formally expressed as

$$H_0: G_{ij}(z) + G_{ij}(-z) = 1 \quad \text{and} \quad G_{ij}(z) = G_{i'j'}(z) \quad \text{for any two pairs } (i, j) \text{ and } (i', j') .$$

In [7] Mehra and in [8] Mehra and Puri had proposed and investigated a family of rank-order tests for the above problem based on a generalization of the Wilcoxon-one-sample ranking procedure: Let $R_{N,l}^{(i,j)}$ denote the rank of $|Z_l^{(i,j)}|$ when the $N = \sum_{i=1}^{K-1} \sum_{j>i} N_{ij}$ absolute values of the observed differences $Z_l^{(i,j)}$, $l = 1, 2, \cdots, N_{ij}$, $(1 \leqq i < j \leqq K)$ are arranged in ascending order of magnitude in a *combined ranking*. For a given set of rank-scores $\xi_{N,\alpha}$, $\alpha = 1, 2, \cdots, N$, define a step function $\xi_N(u)$ over $(0, 1)$, with $\xi_N(u) = \xi_{N,\alpha} = \xi_N(\alpha/(N+1))$ for $(a-1)/N < u \leqq \alpha/N$, $\alpha = 1, 2, \cdots, N$ and set

$$V_N^{(i,j)} = \sum_{l=1}^{N_{ij}} \xi_N(R_{N,l}^{(i,j)}/(N+1)) \operatorname{sign} Z_l^{(i,j)} . \tag{1.1}$$

Assume further the existence of a function $\xi(u)$, $0 < u < 1$, such that

$$\int_0^1 \xi^2(u)\, du < \infty \tag{1.2}$$

and

$$\lim_{N\to\infty} \int_0^1 \{\xi_N(u) - \xi(u)\}^2\, du = 0 . \tag{1.3}$$

For testing the hypothesis H_0, rank-order statistics of the form

$$L_N = L_N(\xi_N, \xi) = \sum_{i=1}^{K} \{\sum_{j\neq i} (V_N^{(i,j)}/(N_{ij})^{\frac{1}{2}})\}^2 \Big/ \left(\frac{1}{N} \sum_{\alpha=1}^{N} \xi_{N,\alpha}^2\right) K \tag{1.4}$$

(with the test consisting in rejecting H_0 when L_N is too large) were considered in [8]. It was shown that if the hypothesis H_0 were true and the conditions (1.2) and (1.3) were satisfied, L_N is distributed in the limit, as $N \to \infty$, as a χ^2-variable with $(K-1)$ df provided $\lim_{N\to\infty} (N_{ij}/N) = \eta_{ij} > 0$ for all (i, j), and that against shift alternatives its asymptotic Pitman-efficiency relative to the normal theory

Received February 4, 1970; revised December 1971.

[1] Prepared with the partial support of the National Research Council of Canada Grant No. A-3061.