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ON THE LENGTH OF THE LONGEST MONOTONE SUBSEQUENCE IN A RANDOM PERMUTATION[1]

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In this short article we prove a concentration result for the length L_n of the longest monotone increasing subsequence of a random permutation of the set $[n] := \{1, 2, \dots, n\}$. It is known (Logan and Shepp [6] and Vershik and Kerov [9]) that

$$\lim_{n \to \infty} \frac{\mathbf{E} L_n}{\sqrt{n}} = 2 \tag{1}$$

but less is known about the concentration of L_n around its mean. Our aim here is to prove the following.

Theorem 1. *Suppose that $\alpha > \frac{1}{3}$. Then there exists $\beta = \beta(\alpha) > 0$ such that for n sufficiently large*

$$\mathbf{Pr}(|L_n - \mathbf{E} L_n| \geq n^{\alpha}) \leq \exp\{-n^{\beta}\}.$$

Our main tool in the proof of this theorem is a simple inequality arising from the theory of martingales. It is often referred to as Azuma's inequality. See Bollobás [2, 3] and McDiarmid [7] for surveys on its use in random graphs, probabilistic analysis of algorithms and so on, and Azuma [1] for the original result. A similar stronger inequality can be read out from Hoeffding [4]. We will use the result in the following form.

Suppose we have a random variable $Z = Z(U)$, $U = (U_1, U_2, \dots, U_m)$, where $U_1, U_2, \dots, U_m$ are chosen independently from probability spaces $\Omega_1, \Omega_2, \dots, \Omega_m$, i.e., $U \in \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$. Assume next that Z does not change by much if U does not change by much. More precisely, write $U \simeq V$ for $U, V \in \Omega$ when U, V differ in at most one component, that is, $|\{i \colon U_i \neq V_i\}| = 1$. We state the inequality we need as a theorem.

Theorem 2. *Suppose Z above satisfies the following inequality:*

$$U \simeq V \quad \textit{implies} \quad |Z(U) - Z(V)| \leq 1,$$

then

$$\mathbf{Pr}(|Z - \mathbf{E}Z| \geq u) \leq 2\exp\left\{-\frac{2u^2}{m}\right\},$$

for any real $u \geq 0$.

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