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CORRECTION NOTE: "STATISTICAL INFERENCE FOR THE MEAN OUTCOME UNDER A POSSIBLY NONUNIQUE OPTIMAL TREATMENT RULE"

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The end of the proof of Lemma 6 in [2] contains an error. Specifically, it is not generally true that $n^{\beta-1}\sum_{j=1}^n R_j \stackrel{d}{=} n^{\beta-1}\sum_{j=1}^n R_j'(\omega')$. As is evidenced by the counterexample in [1], the claim of that lemma does not generally hold unless an additional condition is imposed. Below we state a correction to that lemma that adopts one such condition. Alternative sets of conditions that yield a similar conclusion can be found in [1].

LEMMA 1 (Revision of Lemma 6 in [2]). *Suppose that R_j is some sequence of real-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \nu)$ and such that $j^\beta R_j \xrightarrow{a.s.} 0$ for some $\beta \in [0, 1)$. Then $n^{\beta-1}\sum_{j=1}^n R_j \xrightarrow{a.s.} 0$, and so $n^{-1}\sum_{j=1}^n R_j = o_p(n^{-\beta})$.*

PROOF. Let $\tilde{\Omega} \triangleq \{\omega \in \Omega : \lim_{j\to\infty} j^\beta R_j(\omega) = 0\}$. Fix $\epsilon > 0$ and let $\omega \in \tilde{\Omega}$. There exists some N that, for all $n \geq N$, $n^\beta |R_n(\omega)| < \frac{(1-\beta)\epsilon}{2}$. Also,

$$\frac{1}{n^{1-\beta}}\sum_{j=1}^n j^{-\beta} \leq \frac{1}{n^{1-\beta}}\int_1^n (j-1)^{-\beta}\, dj = \frac{1}{1-\beta}.$$

Hence, for all $n \geq N$,

$$\begin{aligned}
\frac{1}{n^{1-\beta}}\sum_{j=1}^n |R_j(\omega)| &= \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1} |R_j(\omega)| + \frac{1}{n^{1-\beta}}\sum_{j=N}^n \frac{1}{j^\beta} j^\beta |R_j(\omega)| \\
&< \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1} |R_j(\omega)| + \frac{(1-\beta)\epsilon}{2n^{1-\beta}}\sum_{j=N}^n \frac{1}{j^\beta} \\
&\leq \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1} |R_j(\omega)| + \frac{\epsilon}{2}.
\end{aligned}$$

It follows that $\frac{1}{n^{1-\beta}}\sum_{j=1}^n |R_j(\omega)| < \epsilon$ for all n large enough, and thus that $n^{\beta-1}\sum_{j=1}^n R_j(\omega)$ converges to zero as $n \to \infty$. As $\omega \in \tilde{\Omega}$ was arbitrary and $\nu(\tilde{\Omega}) = 1$, $n^{\beta-1}\sum_{j=1}^n R_j$ converges to zero almost surely and, therefore, also in probability. □

The following statements in [2] built on Lemma 6 and, therefore, must be modified to account for the above revision of this lemma:

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