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Topics in loop measures and the loop-erased walk*

Gregory F. Lawler

Department of Mathematics
University of Chicago
Chicago, IL 60637-1546
e-mail: lawler@math.uchicago.edu

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1. Introduction

This is a collection of notes based on a course that I gave at the University of Chicago in fall, 2016 on "Loop measures and the loop-erased random walk". This was not intended to be a comprehensive view of the topics but instead a personal selection of some key ideas including some recent results. This course was to be followed by a course on the Schramm-Loewner evolution (SLE) so there was some emphasis on the ideas that have become important in the study of conformally invariant processes. I will first give some history of the main results I discuss here; this can be considered a personal perspective of the developments of (some of the) ideas. I will follow with a summary of the topics in this paper.

1.1. Some history

I started looking at the loop-erased random walk in my thesis [10] spurred by a suggestion by my advisor, Ed Nelson. My original motivation had been to try to understand the self-avoiding walk. Soon in the investigation, I found out two things: the bad news was that this process was different than the self-avoiding walk, but the good news was that it was a very interesting process with many of the attributes of other models in critical phenomena. In particular, there was an upper critical dimension (in this case $d = 4$) and (conjecturally) conformal invariance in two dimensions. My thesis handled the easiest case $d > 4$. The four dimensional case was significantly harder; I did not have the chance to discuss this in this course even though there is some recent work on the subject [19].

The interest in loop-erased random walk increased when the relationship between it and uniform spanning trees was discovered [23, 25]. I believe there were several independent discoveries of this; one thing I know is that I was not one of the people involved! I found out about it from Robin Pemantle who was trying to construct the infinite spanning tree and forest. He was able to use my results combined with the Aldous-Broder algorithm to show that the limit of the uniform spanning tree was a tree for $d = 4$ and a forest for $d > 4$. I discuss

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