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Rejoinder*

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1 Introduction

We thank Bruno Sansó and the BA editorial team for encouraging discussion on our paper, and the discussants for their interesting and valuable contributions. Our rejoinder is divided into six sections, which provide insight and clarification on the subjects that were raised by the discussants. The paper by Chkrebtii et al. (2016a) and the proposed formalism will hereafter be referred to as UQDE (uncertainty quantification for differential equations).

2 Uncertainty quantification for dynamical systems with the Markov property

Our probabilistic approach to modeling uncertainty in the unknown solution of a dynamical system is motivated by its Markov structure. For example, consider the initial value problem $du/dt = f(t, u)$ on $t \in [0, L]$ with initial condition $u(0) = u_0$. It can be shown that for $t_1 < t_2$, the solution $u(t_2)$ is a function of $u(t_1)$ that does not depend on $u(\tau), \tau \in [0, t_1)$ (e.g., Jazwinski, 1970). Thus, defining probability measures sequentially on a filtration of σ-algebras is a key feature of our proposal and an important distinction with the work of Skilling (1991). Such sequential probability models are also used in simulation of stochastic differential equation (SDE) sample paths, suggesting a relationship with the SDE literature, as described in the insightful discussion of Lysy (2016).

The Markov property is also relevant to the comment of Dass (2016). Equation (2) of UQDE expresses the probabilistic solution $[u, u_t \mid \theta, \Psi, N]$ as a continuous mixture of Gaussian processes obtained by marginalizing $[u, u_t, \mid \mathrm{f}_1, \ldots, \mathrm{f}_N, \theta, \Psi, N]$ over trajectories $\mathrm{f}_1, \ldots, \mathrm{f}_N$ with mixture weights $p(\mathrm{f}_1, \ldots, \mathrm{f}_N)$. Algorithm 1 samples from this mixture by effectively selecting a mixture component from $p(\mathrm{f}_1, \ldots, \mathrm{f}_N)$ and then drawing a sample from $[u, u_t \mid \mathrm{f}_1, \ldots, \mathrm{f}_N]$. However, the Markov structure of the solution u prevents conditioning the trajectory directly on samples from multiple sample paths simultaneously.

Cockayne (2016) suggests a different perspective on this problem. Instead of estimating $u : [0, L] \to \mathbb{R}^p$ given a known vector field function $f(t, \cdot) : \mathbb{R}^p \to \mathbb{R}^p$, the discussant considers estimation of the function $f(\cdot, u(\cdot)) : [0, L] \to \mathbb{R}^p$ directly. In both cases the

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