Statistical Science
2015, Vol. 30, No. 2, 167–169
DOI: 10.1214/15-STS516
Main article DOI: 10.1214/14-STS487

On the Flexibility of Multivariate Covariance Models: Comment on the Paper by Genton and Kleiber

Moreno Bevilacqua, Amanda S. Hering and Emilio Porcu

INTRODUCTION

We congratulate the authors for their considerable effort to collect and synthesize all of the information contained in this review paper. Given the breadth of models, we were particularly inspired by the idea of how a practitioner would choose among them. We define some general criteria of flexibility that should be considered when choosing between different multivariate covariance models, and we apply these criteria in the comparison between the bivariate linear model of coregionalization (LMC) and the bivariate multivariate Matérn.

Which Model Is the Most Flexible?

Since most of the contributions listed by the authors refer to parametric models of multivariate covariances, we seek to answer the question, "which parametric model is more flexible?" We propose to define flexibility with respect to the following:

(A) the colocated correlation coefficient, and
(B) the strength of spatial dependence. For instance, how different can the scales of the cross-covariances and the marginal covariances between the two models be.

As far as (A) is concerned, ideally the colocated correlation coefficient should be defined over the interval $[-1, 1]$. Let us consider models of the type

$$\mathbf{C}(\mathbf{h}) = \left[\sigma_i \sigma_j \rho_{ij} R(\mathbf{h}; \boldsymbol{\theta}_{ij})\right]_{i,j=1}^{2}, \qquad \mathbf{h} \in \mathbb{R}^d, \tag{1}$$

with $R(\cdot)$ being a parametric univariate correlation model in $\mathbb{R}^d$ and $\boldsymbol{\theta}_{ij} \in A \subset \mathbb{R}^q$ being parameter vectors. Here $\sigma_i^2 > 0$, $i = 1, 2$ are the marginal variances, and ρ_{12} is the colocated parameter describing the correlation between the components of the bivariate random field at $\mathbf{h} = 0$. The bivariate Matérn [2] and Wendland [1] models are special cases of equation (1).

For the bivariate Matérn case, the validity bound for ρ_{12} is given in Theorem 3 of [2], and in general it depends on the smoothness parameters, $\boldsymbol{\nu} = (\nu_{11}, \nu_{22}, \nu_{12})'$, and the scale parameters, $\boldsymbol{\alpha} = (\alpha_{11}, \alpha_{22}, \alpha_{12})'$. For instance, assuming a constant smoothness parameter, and $\alpha_{12} < \min(\alpha_{11}, \alpha_{22})$, a necessary and sufficient condition for the validity of the bivariate Matérn becomes $|\rho_{12}| \le \frac{\alpha_{12}^2}{\alpha_{22}\alpha_{11}} \le 1$. In the case where the scale and smoothness parameters are pairwise equal (i.e., the separable case), then $|\rho_{12}| \le 1$, and there are no restrictions on the colocated parameter. These features are also present in the bivariate Wendland construction in [1], where the elements of the matrix-valued covariance are parameterized in the same way as the bivariate Matérn. As the difference between the parameters α_{11} and α_{22} increases, the bound on ρ_{12} becomes tighter, as shown in Figure 1.

The linear model of coregionalization (LMC) does not necessarily share this limitation on the colocated correlation coefficient. In order to illustrate this, we start with a simple example: for the following, we write $R(\cdot) := C(\cdot)/C(0)$, for C some univariate covariance function in $\mathbb{R}^d$. Then, the bivariate LMC correlation model $\mathbf{R}(\mathbf{h}) = [R_{ij}(\mathbf{h})]_{i,j=1}^2$, $\mathbf{h} \in \mathbb{R}^d$, can be written as

$$R_{11}(\mathbf{h}) = a_{11}^2 R_1(\mathbf{h}) + a_{12}^2 R_2(\mathbf{h}),$$
$$R_{22}(\mathbf{h}) = a_{21}^2 R_1(\mathbf{h}) + a_{22}^2 R_2(\mathbf{h}), \quad \text{and}$$
$$R_{12}(\mathbf{h}) = a_{11}a_{21} R_1(\mathbf{h}) + a_{12}a_{22} R_2(\mathbf{h}).$$

The 2×2 matrix $\mathbf{A} = \{a_{ij}\}$ has rank 2. Here we focus, without loss of generality, only on positive cor-

Moreno Bevilacqua is Associate Professor, Department of Statistics, University of Valparaiso, Chile (e-mail: moreno.bevilacqua@uv.cl). Amanda S. Hering is Assistant Professor, Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado 80401, USA (e-mail: ahering@mines.edu). Emilio Porcu is Professor, Department of Mathematics, Technical University Federico Santa Maria, Valparaiso, Chile (e-mail: emilio.porcu@usm.cl).