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Beyond the Valley of the Covariance Function

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1. INTRODUCTION

Multivariate models are under-represented in the literature on spatial statistics. There is a basic reason for this: univariate models are sufficiently complicated to keep us busy. Genton and Kleiber have done a fabulous job compiling and investigating the available models, with a focus on the important class of models that they, with collaborators, introduced. This paper gives a solid state of the art and points out just how many holes there are in the theory and practice associated with these fields. This gives us licence to point out some other holes and to suggest some important directions for the future.

2. THERE IS POWER IN A SPECTRUM

If we were to quibble about one thing in Genton and Kleiber's paper, it would be that we disagree over the extent to which the class of multivariate GRFs has been categorized. Note that this is different from *explicitly* constructing valid cross-covariance functions! To wit, if a multivariate GRF has a spectral representation, the spectral representation given in Section 1.2 completely characterizes the class of stationary multivariate random fields that admit an absolutely continuous spectral measure. This represents a large chunk of interesting GRFs. We note that the paper, by restricting the cross-spectral densities to be real, implicitly assumes that $C_{ij}(\mathbf{h}) = C_{ij}(-\mathbf{h})$, when the minimal necessary requirement is only that $C_{ij}(\mathbf{h}) = C_{ji}(-\mathbf{h})$, which allows for phase differences between the model components. The representation can then be employed constructively as follows. Let $\boldsymbol{\omega} \to \mathbf{S}(\boldsymbol{\omega})$ be a mapping from $\mathbb{R}^d$ to the set of Hermitian nonnegative definite matrices, the elements of which the cross-spectral densities, denoted f_{ij} in the paper, are here subject to $f_{ij}(\boldsymbol{\omega}) = \overline{f_{ji}(\boldsymbol{\omega})}$. Then, for any complex, matrix-valued function $\mathbf{L}(\boldsymbol{\omega})$ such that $L_{ij}(\boldsymbol{\omega}) = \overline{L_{ij}(-\boldsymbol{\omega})}$ and $\mathbf{S}(\cdot) = \mathbf{L}(\cdot)\overline{\mathbf{L}(\cdot)}$,

$$
\begin{aligned}
\mathbf{x}(\mathbf{s}) &= \int_{\mathbb{R}^d} \mathbf{L}(\boldsymbol{\omega}) e^{i\mathbf{s}\cdot\boldsymbol{\omega}}\, d\widetilde{\mathbf{W}}(\boldsymbol{\omega}) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{L}(\boldsymbol{\omega}) e^{i(\mathbf{s}-\mathbf{s}')\cdot\boldsymbol{\omega}}\, d\boldsymbol{\omega}\, d\mathbf{W}(\mathbf{s}'),
\end{aligned} \tag{1}
$$

where $d\widetilde{\mathbf{W}}(\cdot) \in \mathbb{C}^p$ and $d\mathbf{W}(\cdot) \in \mathbb{R}^p$ are Gaussian white noise processes on $\mathbb{R}^d$ understood as random measures with $d\widetilde{W}_i(\boldsymbol{\omega}) = \overline{d\widetilde{W}_i(-\boldsymbol{\omega})}$, $\mathbb{E}[d\widetilde{\mathbf{W}}(\boldsymbol{\omega}) \cdot \overline{d\widetilde{\mathbf{W}}(\boldsymbol{\omega}')}] = \delta(\boldsymbol{\omega} - \boldsymbol{\omega}')\mathbf{I}\, d\boldsymbol{\omega}$, and $\mathbb{E}[d\mathbf{W}(\mathbf{s})\overline{d\mathbf{W}(\mathbf{s}')}] = \delta(\mathbf{s}-\mathbf{s}')\mathbf{I}\, d\mathbf{s}$ (Adler and Taylor, 2007; Lindgren, 2012). This representation only covers multivariate GRFs with absolutely continuous spectral measures; however, the same procedure applies to fields with an atomic spectral representation. The abstract feature that is hiding in all of this specificity is that we are explicitly constructing a square root of the multivariate covariance operator and using this square root to filter the multivariate white noise. On a compact domain, the covariance operator is a compact, trace class operator, and so this square root is well defined using the usual functional calculus.

Another reason to further emphasize this spectral representation is that it is not only constructive in its own right, but also useful when transformed back to the nonspectral domain. Kernel convolution methods (Higdon, 1998) have a storied history in univariate spatial statistics and their generalization to the multivariate case is straightforward (Simpson, Lindgren and Rue, 2012; Bolin and Lindgren, 2013). Their advantage is that it is never necessary to identify the spectrum of the process or, in fact, the cross-covariance structure. Rather, for any L^2 matrix-valued function $\mathbf{K}(\cdot,\cdot)$, $\mathbf{x}(s) = \int \mathbf{K}(\mathbf{s},\mathbf{s}')\, d\mathbf{W}(\mathbf{s}')$ is a valid mutlivariate GRF, which can be approximated by (carefully) approximating the corresponding integral with a sum. In

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