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ACE Bounds; SEMs with Equilibrium Conditions

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We congratulate the author on an enlightening account of the instrumental variable approach from the viewpoint of Econometrics. We first make some comments regarding the bounds on the ACE under the nonparametric IV model, and then discuss potential outcomes in the market equilibrium model.

1. ACE BOUNDS UNDER THE IV MODEL

We consider the model in which X and Y are binary, taking values in $\{0, 1\}$, while Z takes K states $\{1, \dots, K\}$. We use the notation $X(z_i)$ to indicate $X(z = i)$, similarly $Y(x_j)$ for $Y(x = j)$. We consider four different sets of assumptions:

(i) $Z \perp\!\!\!\perp Y(x_0), Y(x_1), X(z_1), \dots, X(z_K)$;
(ii) $Z \perp\!\!\!\perp Y(x_0), Y(x_1)$;
(iii) for $i \in \{1, \dots, K\}$, $j \in \{0, 1\}$, $Z \perp\!\!\!\perp X(z_i), Y(x_j)$;
(iv) there exists a U such that $U \perp\!\!\!\perp Z$ and for $j \in \{0, 1\}$, $Y(x_j) \perp\!\!\!\perp X, Z \mid U$.

Condition (i) is joint independence of Z and all potential outcomes for Y and X. (ii) does not assume independence (or existence) of counterfactuals for X. (iii) is a subset of the independences in (i), none of which involve potential outcomes from different worlds.[1] The counterfactual independencies (i), (ii), (iii) arise most naturally in the context where the instrument is randomized, as depicted by the DAG in Figure 1(a). Assumption (iii) may be read (via d-separation) from the Single-World Intervention Graph (SWIG)[2] $\mathcal{G}_1(z, x)$, depicted in Figure 1(b), which represents the factorization of $P(Z, X(z), Y(x), U)$, implied by the IV model.

Lastly (iv) consists of only three independence statements, but does assume the existence of an unobserved variable U that is sufficient to control for confounding between X and Y. No assumption is made concerning the existence of counterfactuals $X(z)$; confounding variables (U^*) between Z and X are permitted (so long as $U^* \perp\!\!\!\perp U$). The DAG $\mathcal{G}_2$ and corresponding SWIG $\mathcal{G}_2(x)$ are shown in Figure 1(c), (d). In Richardson and Robins (2014), we prove the following.

THEOREM 1. *Under any of the assumptions* (i), (ii), (iii), (iv), *the set of possible joint distributions* $P(Y(x_0), Y(x_1))$ *are characterized by the* $8K$ *inequalities*:

$$P\bigl(Y(x_i) = y\bigr) \le P(Y = y, X = i \mid Z = z) + P(X = 1 - i \mid Z = z), \tag{1}$$

$$P\bigl(Y(x_0) = y, Y(x_1) = \tilde{y}\bigr) \le P(Y = y, X = 0 \mid Z = z) + P(Y = \tilde{y}, X = 1 \mid Z = z). \tag{2}$$

Thus a distribution $P(X, Y \mid Z)$ is compatible with the stated assumptions if and only if there exists a distribution $P(Y(x_0), Y(x_1))$ satisfying (1) and (2).

THEOREM 2. *Under any of the assumptions* (i), (ii), (iii), (iv) *for all* $i, j \in \{0, 1\}$, $P(Y(x_i) = j) \le g(i, j)$, *where*

$$\begin{aligned} g(i, j) \equiv \min\Bigl\{ &\min_z \bigl[P(X = i, Y = j \mid Z = z) + P(X = 1 - i \mid Z = z)\bigr], \\ &\min_{z, \tilde{z}: z \ne \tilde{z}} \bigl[P(X = i, Y = j \mid Z = z) + P(X = 1 - i, Y = 0 \mid Z = z) \\ &\quad + P(X = i, Y = j \mid Z = \tilde{z}) + P(X = 1 - i, Y = 1 \mid Z = \tilde{z})\bigr]\Bigr\}. \end{aligned}$$

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[1] In other words, they do not involve both $Y(x_0)$ and $Y(x_1)$, nor $X(z_i)$ and $X(z_j)$ for $i \ne j$.

[2] See Richardson and Robins (2013) for details.