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Discussion of "Estimating Random Effects via Adjustment for Density Maximization" by C. Morris and R. Tang

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We congratulate Morris and Tang for an interesting addition to empirical Bayes methods, and for tackling a difficult and nagging problem in variance estimation. The ADM adjustment appears to bring on interesting properties, not just in variance estimation but also in estimation of the means. In this discussion we want to focus on the latter topic, and see how the ADM-derived estimators of a normal mean perform in a decision-theoretic way. To facilitate this we will stay with the simple model

$$y_i|\theta_i \sim N(\theta_i, V), \quad \theta_i \sim N(0, A). \tag{1}$$

1. THE JAMES–STEIN ESTIMATOR AS GENERALIZED BAYES (NOT!)

We first address the comment of Morris and Tang in Section 2.5, that the prior $A \sim \text{Unif}(0, \infty)$ is strongly suggested because the James–Stein estimator is the posterior mean if we take $A \sim \text{Unif}(-V, \infty)$. Professor Morris has noted this before, and in the interest of understanding, we want to show this calculation and comment on its relevance.

Writing $\mathbf{y} = (y_1, \dots, y_k)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$, the posterior expected loss from model (1), with the $A \sim \text{Unif}(-V, \infty)$ prior, is

$$\int_{-V}^{\infty} \int_{\Re^p} |\boldsymbol{\theta} - \delta(\mathbf{y})|^2 \cdot \frac{e^{-|\mathbf{y}-\theta|^2/(2V)}}{(2\pi V)^{k/2}} \frac{e^{-|\theta|^2/(2A)}}{(2\pi A)^{k/2}} d\boldsymbol{\theta}\, dA, \tag{2}$$

and factoring the exponent in (2) and writing $B = V/(V + A)$ shows that

$$\boldsymbol{\theta}|\mathbf{y}, A \sim N\big((1 - B)\mathbf{y}, V(1 - B)\big),$$

$$A|\mathbf{y} \sim \left(\frac{1}{V + A}\right)^{k/2} e^{-(1/(2(V+A)))|\mathbf{y}|^2}.$$

The Bayes rules is the posterior mean, which we can calculate as

$$\begin{aligned} \mathrm{E}(\boldsymbol{\theta}|\mathbf{y}) &= \mathrm{E}[\mathrm{E}(\boldsymbol{\theta}|\mathbf{y}, A)] \\ &= \mathrm{E}[(1 - B)\mathbf{y}|\mathbf{y}] = [1 - \mathrm{E}(B|\mathbf{y})]\mathbf{y}. \end{aligned}$$

We now, very carefully, calculate $\mathrm{E}(B|\mathbf{y})$, yielding

$$\begin{aligned} \mathrm{E}(B|\mathbf{y}) &\propto \int_{-V}^{\infty} \left(\frac{V}{V + A}\right)\left(\frac{1}{V + A}\right)^{k/2} \cdot e^{-(1/(2(V+A)))|\mathbf{y}|^2}\, dA \\ &= V \int_{1/V}^{\infty} t^{k/2-1} e^{-t|\mathbf{y}|^2/2}\, dt \\ &\quad + V \int_{0}^{1/V} t^{k/2-1} e^{-t|\mathbf{y}|^2/2}\, dt, \end{aligned}$$

where we make the transformation $t = 1/(V + A)$, with the first integral coming from $A \in (-V, 0)$. Noting that the integrand is the kernel of a chi-squared density, we finally have

$$\begin{aligned} \mathrm{E}(B|\mathbf{y}) \propto \frac{V\Gamma(k/2)2^{k/2}}{(|\mathbf{y}|^2)^{k/2}} [P(\chi_k^2 \geq |\mathbf{y}|^2/V) \\ + P(\chi_k^2 \leq |\mathbf{y}|^2/V)], \end{aligned} \tag{3}$$

where χ_k^2 is a chi-squared random variable with k degrees of freedom. Since the chi-squared probabilities sum to 1, normalizing this expectation (dividing by $\frac{\Gamma(k/2-1)2^{k/2-1}}{(|\mathbf{y}|^2)^{k/2-1}}$) results in $\mathrm{E}(B|\mathbf{y}) = V(k - 2)/|\mathbf{y}|^2$, yielding the James–Stein estimator. There are a number of things to note:

1. If this were a valid calculation, it would contradict such important papers as Brown (1971) and Strawderman and Cohen (1971), which provided complete characterizations of admissible generalized Bayes estimators.

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