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DISCUSSION OF: BROWNIAN DISTANCE COVARIANCE

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Concepts of dependence are central in the theory of statistics and to most of its applications. It is therefore a pleasure to commend the authors—hereafter SR—for their theoretical contributions to our understanding of some of its subtler aspects, and for their provocatively interesting data analysis examples.

Standard measures, such as the product moment correlation, Spearman's rank correlation, Kendall's tau or Fisher–Yates' normal scores statistic, are all deficient. These only measure dependence of a "monotone character" and will not be effective even in such simple situations as when Y has a nonmonotone regression on X and X is sampled randomly. Another simple example where such measures fail is when $X_i = V_i Z_i$ and $Y_i = V_i Z_i'$, where the "innovations" Z_i, Z_i' are, say, independent standard normal variates, but the X_i, Y_i share a common random scaling V_i; such structures arise in the stochastic volatility models of finance.

Owing to their importance, consistent measures of dependence—and, in particular, measures which in principle admit sample analogues on the basis of which tests consistent against all dependence alternatives can be constructed—have appeared previously, and at least as far back as Renyi (1953). Renyi's measure has, of course, ideal theoretical properties, but implementing its sample analogues is not straightforward, and for that reason it has not become a mainstay in applications. [See, e.g., Buja (1990).] In that respect the dependence measure (which predate's Renyi's) introduced by Hoeffding (1948), and later rediscovered in a more transparent form by Blum, Keifer and Rosenblatt (1961), has been more successful. See also Csörgő (1985).

There is also some precedent for the measures proposed at (2.4) and (2.6) in SR (at least for the case when $\alpha = 1$), although these appear here in a substantially extended form, and based on a novel approach with fresh interpretations. For example, Feuerverger (1993)—hereafter F93—proposed measures based on

$$\int\int \frac{|f^n_{X,Y}(s,t) - f^n_X(s) f^n_Y(t)|^2}{(1-e^{-s^2})(1-e^{-t^2})} W(s,t)\, ds\, dt, \tag{1}$$

with $W(s,t)$ a suitable weight function. In F93, the denominator term in (1) was suggested on the basis of its being (proportional to) the limiting variance of the term within the modulus in the numerator, under the null hypothesis of independence in the case of standard normality. The ratio within the integral in (1) is