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Monomial ideals whose depth function has any given number of strict local maxima

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In recent years there have been several publications concerning the stable set of prime ideals of a monomial ideal, see for example [4], [7], [10] and [9]. It is known by Brodmann [2] that for any graded ideal I in the polynomial ring S (or any proper ideal I in a local ring) there exists an integer k_0 such that $\operatorname{Ass}(I^k)=\operatorname{Ass}(I^{k+1})$ for $k\geq k_0$. The smallest integer k_0 with this property is called the *index of stability* of I and $\operatorname{Ass}(I^{k_0})$ is the set of *stable prime ideals* of I. A prime ideal $P\in\bigcup_{k=1}^{\infty}\operatorname{Ass}(I^k)$ is said to be *persistent* with respect to I if whenever $P\in\operatorname{Ass}(I^k)$ then $P\in\operatorname{Ass}(I^{k+1})$, and the ideal I is said to satisfy the *persistence property* if all prime ideals $P\in\bigcup_{k=1}^{\infty}\operatorname{Ass}(I^k)$ are persistent. It is an open question (see [6] and [13, Question 3.28]) whether any square-free monomial ideal satisfies the persistence property.

We call the numerical function $f(k)=\operatorname{depth}(S/I^k)$ the *depth function* of I. It is easy to see that a monomial ideal I satisfies the persistence property if all monomial localizations of I have non-increasing depth functions. In view of the above mentioned open question it is natural to ask whether all square-free monomial ideals have non-increasing depth functions. The situation for non-square-free monomial ideals is completely different. Indeed, in [8, Theorem 4.1] it is shown that for any non-decreasing numerical function f, which is eventually constant, there exists a monomial ideal I such that $f(k)=\operatorname{depth}(S/I^k)$ for all k. Note that a similar result for non-increasing depth functions is not known, even though it is expected that all square-free monomial ideals have non-increasing depth functions. In general the depth function of a monomial ideal does not need to be monotone. Examples of monomial ideals with non-monotone depth functions are given in [12, Example 4.18] and [8]. The question arises which numerical functions are depth functions of monomial ideals. Since $\operatorname{depth}(S/I^k)$ is constant for all $k\gg 0$ (see [1]), any depth function is eventually constant. So the most wild conjecture one could make is that any numerical function which is eventually constant is indeed the depth function of a monomial ideal. In support of this conjecture we show in our theorem that for any given number n there exists a monomial ideal whose depth function has precisely