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Holomorphic functions and Hausdorff dimension

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1. Introduction

Let $D(z, r)$ represent the open disc with center z and radius r, and let $C(z, r)$ represent its boundary oriented in the usual counter-clockwise manner. We define the class A_α, $1 \leqslant \alpha \leqslant 2$, as follows:

$f(z)$ is in the class A_α if

(i) $f(z)$ is a continuous complex-valued function defined in $D(0, 1)$, and

(ii) there exist a constant K and a $\gamma > \alpha$ such that for $0 < \varrho < 1$ and $0 < r < 1 - \varrho$

$$\int_{D(0,\varrho)} \left| \int_{C(z,r)} f(\zeta)\, d\zeta \right|^2 dx\, dy \leqslant K r^{2+\gamma}.$$

We define the class B_α in the same manner as the class A_α except in (ii), we only require that $\gamma \geqslant \alpha$. It is clear that the class B_α is the natural widening of the class A_α.

We shall say that the relatively closed set $E \subset D(0, 1)$ [i.e. the complement of E in $D(0, 1)$ is open] is a removable set for the class A_α if the following fact holds: *If f is in A_α and f is holomorphic in $D(0, 1) \sim E$, then f is holomorphic in $D(0, 1)$.* E is a removable set for the class B_α is defined in a similar manner.

In this paper, we intend to establish the following result:

Theorem. *A necessary and sufficient condition that a relatively closed set E contained in $D(0, 1)$ be a removable set for the class A_α, $1 \leqslant \alpha \leqslant 2$, is that the Hausdorff dimension of E be $\leqslant \alpha$. Furthermore, the sufficiency condition is in a certain sense best possible, i.e., it is false for the class B_α.*

If $\alpha < 1$ and the Hausdorff dimension of $E \leqslant \alpha$, Besicovitch has shown that E is a removable set for the class of continuous functions in $D(0, 1)$. He has shown even more, namely that if E is a countable union of sets of finite length, then E is a removable set for this last named class of functions. For the details of his result see either [7, p. 197] or [1].

We next note that the sufficiency of the above theorem in the special case $\alpha = 2$ is essentially known already and is a corollary of [10; Theorem 1, p. 76].

2. Proof of the necessary condition

We first establish the necessary condition of the above theorem.

Since every set contained in $D(0, 1)$ is of Hausdorff dimension $\leqslant 2$, it follows that if E is a removable set for the class A_2 then E is of Hausdorff dimension $\leqslant 2$.