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Polar sets and removable singularities of partial differential equations

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0. Introduction

The question of removable singularities for partial differential equations is essentially the following: If u is a solution of such an equation in a domain $V \subset R^n$ with a closet set S (of measure zero) removed; and if u is assumed to belong to a class limiting its size near S (for example $u \in L_p$), what conditions can be put on the size of S to insure that u (after being redefined in S) is a solution in all of V? For example, every bounded harmonic function in a punctured disk has a removable singularity at the "puncture". Here the class limiting the size of u may be taken to be the class of bounded functions; and S may be taken to be a single point.

L. Carleson [4] has shown that if u is harmonic in $V-S$ (V a bounded n-dimensional domain, S a compact set $\subset V$) and S has finite $n-2p'$ dimensional Hausdorff measure $(1/p+1/p'=1)$ then the singularities of u on S are removable provided $u \in L_p$. Serrin [10] has extended this result to second order linear elliptic equations with Hölder continuous coefficients, and has given a different sufficient condition for second order (linear or quasilinear) elliptic equations [11].

Our aim is to treat linear equations of arbitrary order. (Some results of this nature are contained in [3]). We begin by observing (in section 1) that the question of removable singularities of solutions in L_p is closely tied to the notion of "$m-p$ polar" sets in R^n (A compact set S is $m-p$ polar if every element in $H_{-m,p'}(R^n)$ with support in S vanishes), a notion apparently first introduced by Hörmander and Lions [6]. The relationships between the two concepts is expressed in theorems 1 and 2. These theorems are proved in sections 1 and 2 where they are applied to second order equations.

In section 3 generalizations of the $H_{m,p}$ spaces called "A spaces" are introduced, and, using these, sharper results are obtained for equations which are assumed to be of a more special form. Section 4 deals with geometric sufficient conditions for $m-p$ polarity of sets, which seems to be of interest independently of the question of removable singularities (see for example [5]); while in section 5 similar results are obtained for the A-spaces introduced earlier. Finally, as an illustration, the latter results are applied to give geometric conditions for removability of singularities of the heat equation.

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